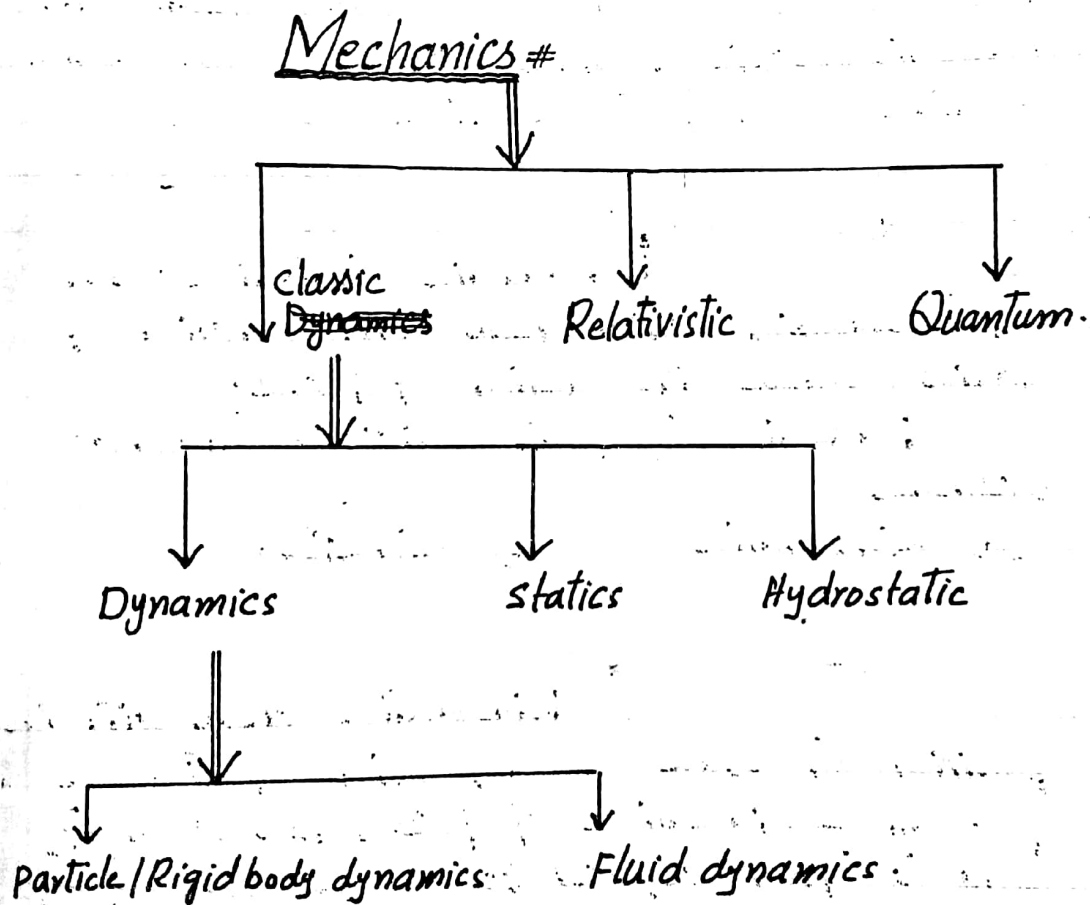


Mechanics- P-I

MECHANICS

Outlines of Syllabus

General motion of a rigid body, Euler's Theorem and Chasles' Theorem, Euler's angles, Moments and products of Inertia, Inertia tensor, Principal axes and Principal Moments of inertia, Kinetic energy, and Angular Momentum of a rigid body, Momental ellipsoid and equimomental systems, Euler's dynamical equations and their solution in special cases. Heavy symmetrical top, equilibrium of a rigid body, general conditions of equilibrium and deduction of conditions in special cases.



③

Mechanics #

Mechanics is the science concerning to the motion and rest and laws which govern these states of bodies or particles.

Classical Mechanics #

That branch of mechanics which deals with macroscopic objects.

Relativistic Mechanics #

In this branch of mechanics the velocity of light is considered absolute and all velocities are taken relative to the velocity of light.

Quantum Mechanics #

It deals with the microscopic particles or bodies (fundamental particles or elementary particles)

Dynamics #

Dynamics is that branch of Mechanics which deals with motion of bodies under the action of forces.

Dynamics is further divided into two branches

(a) Kinematics

(b) Kinetics.

Kinematics #

Kinematics deals with the position in space as a function of time apart from all considerations of force, mass or energy and is often referred to as the "geometry of

3

of motion. The calculation of flight trajectories of aircraft, spaceships, rockets are examples of kinematical problems.

Kinetics

That branch of dynamics which deals with effects of forces on the motion of bodies and links or relates the action of forces on bodies to their resulting motion.

Remarks # The functions involved in dynamical problems are (for the most part) differential co-efficients w.r.t time t , as independent variable. The formulation of a dynamical problem in general consists of one or more relations between certain variables (Co-ordinates of position) and their differential co-efficients w.r.t time. Such relations are called differential equations.

Particle

A body with negligible size so that it may be described by a geometrical point is called particle.

OR

A very very small mass concentrated at a point is called particle.

OR

A very small mass which can not be further divided.

OR

In mathematical sense a particle is a body whose dimensions approach zero so that it may be analysed as a point mass.

4

Frequently a particle is chosen as a differential element of a body. Also when the dimensions of a body are irrelevant to the description of its position or its motion, the body may be treated as particle.

Any body whose size is very small as compared to other bodies can be regarded as a particle.

Rigid Body

A body is said to be rigid when it has no relative deformation between its parts under external forces. In other words the distance between the particles of a rigid body does not change.

In real world no rigid body exist and therefore the concept of rigid body is an ideal concept.

All real objects undergo some changes in shape when they are subjected to forces. When such changes in shape are negligible as compared to overall dimension of body or with the changes in the position of body, then the body may be considered as a rigid body.

Remarks #1) When a body moves all parts of the body have not necessarily the same motion but if the body is very small, then the differences between the motions of its different parts are unimportant.

(2) In order to describe the motion of body or of a point two things are needed.

(a) a frame of reference (b) a time-keeper.

Reference Frame

Position in space is determined relative to some geometric reference system by means of linear and angular measurements.

The basic frame of reference for laws of Newtonian mechanics is the primary inertial system or astronomical frame of reference which is an imaginary set of rectangular axes assumed to have no translation or rotation in space. Measurements show that laws of Newtonian mechanics are valid for this reference system as long as any velocities involved are negligible compared with speed of light.

The inertial frame is required i.e. frame be such that a free particle in it continues to be in state of rest or of uniform motion along a straight line. The reason for employing inertial frames is that physical laws remain unaffected by the choice of the frame of reference.

The search for such frame, call it the absolute frame, w.r.t. which other inertial frames are to be defined, is, therefore important.

Existence of an Absolute Frame

Consider a frame of reference fixed on the surface of the earth. Experiments show that most of the physical phenomena can be adequately well explained with reference to this frame. But if we look a little more closely, there are some discrepancies between the predictions made in this frame and the actual observations and a correction to the basic equations of mechanics

must be applied for measurements made relative to the earth's reference frame. For example, assume that a body is falling under the effect of gravity from a height which is small compared with the radius of earth, so that the acc. due to gravity may be taken as constant. The usual equation of motion indicates a vertical fall of the body. It also appears so. But careful observation shows that there is a slight deviation from the vertical to right of its course. The curved wind shows the same phenomenon more clearly. Likewise, a body on which no external force is acting appears to move in the earth's frame of reference with constant speed along a straight line. Careful measurements, however, would show that trajectory is not really a straight line but slightly curved. These considerations show that the frame fixed to earth's surface is not absolute because of the earth's rotation.

Consider now a frame of reference located at the centre of mass of earth and fixed orientation w.r.t to the distant stars which may be assumed to at rest to a very good approximation. The frame fixed to the earth surface is now rotating w.r.t this frame with the angular velocity of earth. Computation of the trajectory of a body falling freely under gravity near the surface of earth now shows the deviation ^{to the} right correctly. This frame is therefore nearer to the inertial frame than the frame attached to the surface. It is not, however, an inertial frame since it does not explain correctly the tidal effect of the moon. The analysis of earth-moon system in this frame indicates that there will be only one tide in ^{every} 24 hours at any

7

, whereas actually two tides are observed every day. This discrepancy is caused by the orbital motion of the earth. Theoretically, the frame attached to the centre of the universe (if we can define one) may be answer to our search for an absolute frame.

Inertial Frame of Reference

An inertial frame of reference is the frame in which Newton's Laws remain valid. Because Newton's 1st and 3rd Laws can be derived from the 2nd law, we can define an inertial frame a frame in which 2nd law of motion remains valid.

Note# (1) Rotating frames are not inertial frame.

(2) In a mathematical model, we may employ a rigid body as the frame of reference. As we may introduce any number of rigid bodies moving relative to one another, therefore we have thus at our disposal any number of frame of reference.

Time

Time is a measure of the succession of events and is considered as absolute quantity in Newtonian mechanics.

Note# It is important to realize that there is no such thing as absolute time. but the period of rotation of the Earth relative to the fixed stars provides a unit of time, the sidereal day, which so far as it can be tested with other time measures is constant and therefore adequate for the purposes of ordinary dynamics.

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Space# space is the geometric region occupied by the bodies.

Particle Dynamics

Although our topic is rigid body dynamics but since the rigid body is considered as made of particles which are closely packed and distance between which remains fixed, therefore it is important to discuss some basic principals of particle dynamics.

Newton's 1st law#

A particle subject to no external force is either at rest or moves with uniform velocity.

Newton's 2nd law#

The acceleration of particle is proportional to the resultant force acting on it and is in the direction of this force.

$$\underline{F = ma}$$

Newton's 3rd law#

The forces of action and reaction between interacting bodies are equal in magnitude, opposite in direction and collinear.

Velocity#

The rate of change of displacement (p.v) is called velocity

$$\underline{V = \frac{dx}{dt}} \rightarrow \textcircled{1}$$

If a particle moves a distance δx in time δt . Then $\frac{\delta x}{\delta t} = \text{average velocity}$

If we take limit when $\delta t \rightarrow 0$, then

$$\lim_{\delta t \rightarrow 0} \frac{\delta x}{\delta t} = \frac{dx}{dt} = \text{Velocity}$$

$$\text{Acc} = a = \frac{dv}{dt}$$

Vector Addition of Forces

Forces acting on a particle obey the laws of vector addition. We state it by two methods

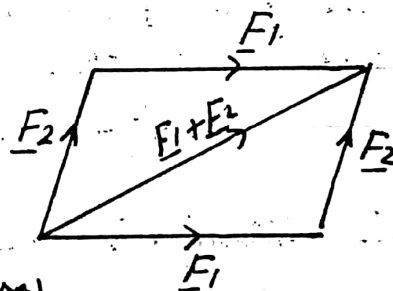
(a) Triangle law #

$$\vec{OA} + \vec{AB} = \vec{OB}$$



(b) # Parallelogram law #

If two forces are represented as the adjacent sides of a parallelogram, then their resultant is represented by the diagonal of the parallelogram. This method is called the parallelogram law.



Types of Vectors

- (a) # Free vector (as usual here)
- (b) # Sliding vector (Like a force on a rigid body which acts along a line)
- (c) # Localized vector (Bounded vector) Like a force on a particle which acts upon a point.

Equilibrium Law

It states that forces are in equilibrium (i.e. their mechanical effect is zero) if

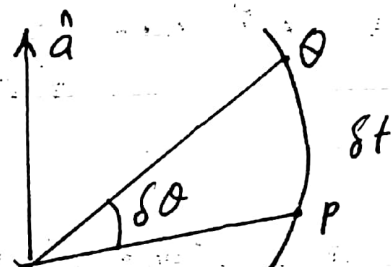
- (i) they are equal in magnitude
- (ii) they are opposite in direction
- (iii) they act upon a same line i.e. their line of action is same

The Principal of Superposition of Forces

The effect of forces acting on a particle or body are mutually independent i.e. the forces act on a particle independently.

Angular Velocity

If a body is at point P and after a short interval of time δt it reaches at point Q



, then the change in angular displacement is $\delta \theta$

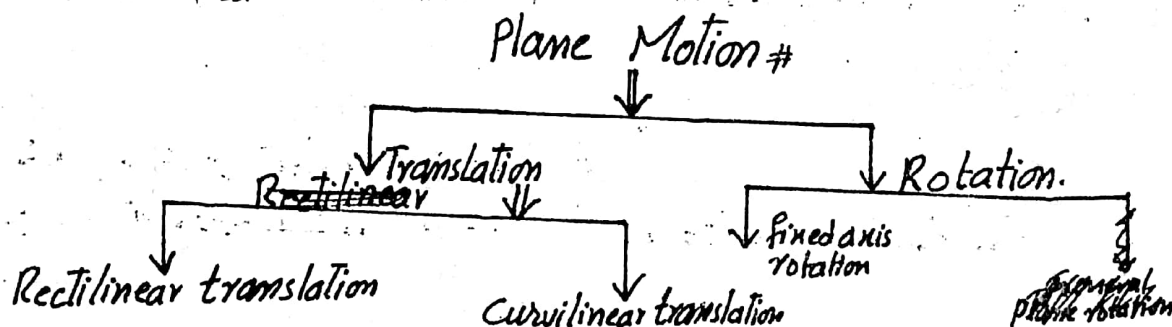
Angular velocity is defined as the rate of change of angular co-ordinate w.r.t. t and is denoted by ω

$$\text{i.e. } \omega = \frac{d\theta}{dt} \hat{a}$$

Its direction is determined by right hand rule

Types of the Plane Motion of a Rigid Body

The plane motion of a rigid body may be divided as



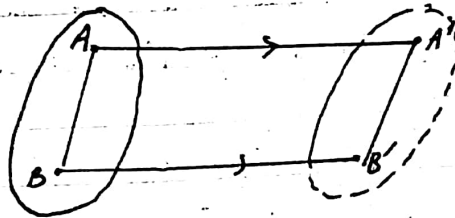
Translation

Any motion in which every line fixed in the body remains parallel to its original position at all times. It is of two types

(a) Rectilinear translation (b) Curvilinear translation

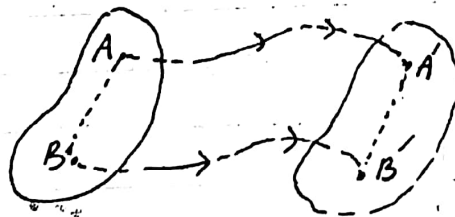
Rectilinear Translation

It is a translation in which all points in the body move in straight lines.



Curvilinear Translation

It is the translation in which all points move on congruent curves. In curvilinear translation there is no rotation of any line in the body.



Remarks # It should be noted that in each case of translation the motion of the body is completely specified by the motion of any point. Since all points have the same motion.

Rotation about Fixed Axis

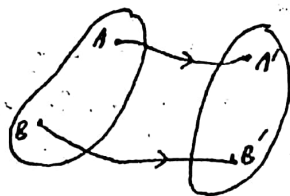
It is the angular motion about the fixed axis. It follows

that all particles move in circular paths about the axis of rotation and all lines in the body (including those that do not pass through the axis) rotate through the same angles in the same time.

Remarks# It should be noted that in all types of above cited motion, all particles in the body move in parallel planes. The motion, however, is represented by its projection onto a single plane parallel to the motion and this plane is called plane of the motion. This plane is usually considered as passing through the centre of mass of the body.

General Plane Motion of Rigid Body#

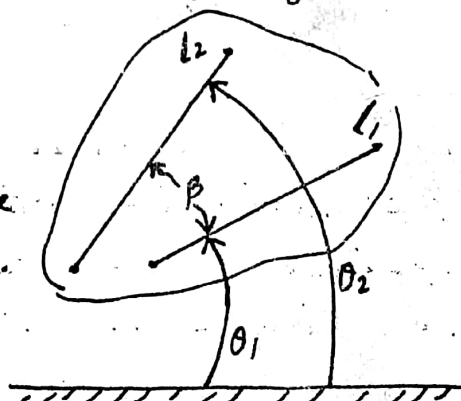
It is a combination of translation and rotation



Problem# Prove that all lines in a rigid body in its plane of motion have the same angular displacement, the same angular velocity and the same angular acceleration

Sol# fig shows a rigid body which undergoes plane motion in the plane of figure.

The angular position of any two lines L_1 & L_2 attached to body are specified by θ_1 and θ_2 measured from some fixed reference direction which is convenient. Since the angle β is invariant



Therefore the relation

$$\theta_2 = \theta_1 + \beta$$

gives

$$\dot{\theta}_2 = \dot{\theta}_1 \rightarrow \textcircled{1}$$

and

$$\ddot{\theta}_2 = \ddot{\theta}_1 \rightarrow \textcircled{2}$$

or during a finite interval

$$\Delta\theta_2 = \Delta\theta_1$$

Thus all lines in a rigid body in its plane of motion have the same angular displacement, the same angular velocity and the same angular acceleration.

Mass Moment of Inertia About an Axis

When a rigid body rotates about a fixed axis then moment of inertia (or sec moment) of i th particle m_i is given by

$$m_i r_i^2$$

where m_i is the mass of i th particle, and r_i is the distance of i th particle from the axis of rotation.

The moment of inertia of the whole body is given by

$$\sum_i m_i r_i^2$$

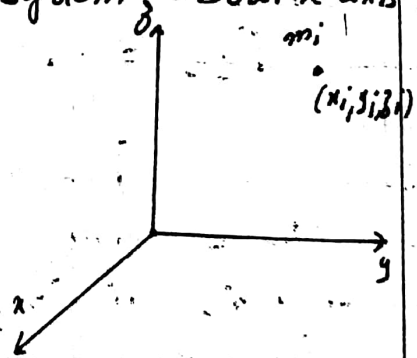
where summation is taken over all the particles of the body.

Moment of Inertia of System about axes

(1) Moment of inertia of a system about x -axis is

$$\sum_i m_i (y_i^2 + z_i^2)$$

Where (x_i, y_i, z_i) are the co-ordinates of m_i w.r.t the co-ordinates system $OXYZ$.



14

Here $\sqrt{y_i^2 + z_i^2}$ is distance of m_i from x -axis which is calculated by distance of (x_i, y_i, z_i) from point $(x_i, 0, 0)$ on x -axis using distance formula

(2) # Moment of inertia of the system about y -axis is

$$\sum_i m_i (x_i^2 + z_i^2)$$

(3) Moment of inertia about z -axis is

$$\sum_i m_i (x_i^2 + y_i^2)$$

Products of Inertia w.r.t Axes

The expressions $\sum m_i y_i z_i$, $\sum m_i z_i x_i$, $\sum m_i x_i y_i$ are called products of inertia of the system about y & z -axis, z & x -axis, x & y -axis.

Note # (1) A moment of inertia is always +ve
(2) A product of inertia may be +ve or -ve

Moment of Inertia for Continuous Distribution of Mass

In case of continuous distribution of mass

$$I_{xx} = \text{Moment of inertia about } x\text{-axis} \\ = \int (y^2 + z^2) dm$$

where dm is the element of mass (mass element) at (x, y, z)

In case of thin uniform rod, density ρ is

and we have

$$dm = \rho dx$$

where dx element of differential length.

In case of flat Lamina

$$dm = \rho dx dy = \rho dA$$

ρ is the density at point (x, y)

In plane polar Co-ordinates

$$dm = \rho r dr d\theta$$

In case of solid sphere

$$dm = \rho dx dy dz = \rho dV$$

In case of spherical polar Co-ordinates

$$dm = \rho r^2 \sin \theta dr d\theta d\phi$$

Polar Moment of Inertia

The sum

$$I_0 = I_x + I_y$$

is called the polar moment of inertia or the moment of inertia w.r.t origin.

First Moments About the Co-ordinates Planes

$$M_{yz} = \iiint_{\text{Region of solid}} x \rho dV, \quad M_{xz} = \iiint_R y \rho dV$$

$$M_{xy} = \iiint_R z \rho dV$$

Centre of Mass

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}$$

where m is total mass

Moments of Inertia (Second Moment)

$$I_x = \iiint_R (y^2 + z^2) \rho \, dV$$

$$I_y = \iiint_R (x^2 + z^2) \rho \, dV$$

$$I_z = \iiint_R (x^2 + y^2) \rho \, dV$$

Moment of Inertia about any line L is

$$I_L = \iiint r^2 \rho \, dV$$

$r(x, y, z)$ = distance from the points (x, y, z) to line L

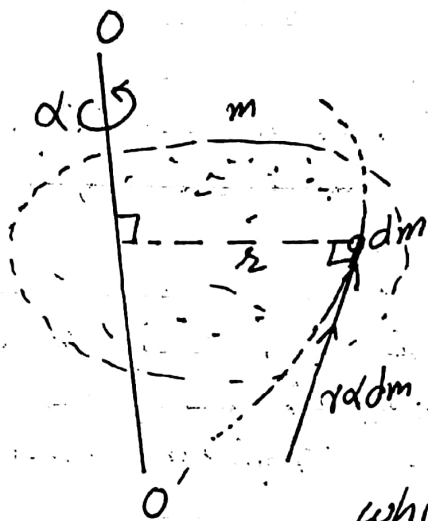
Notations For Moments of Inertia

For plane motion if the motion is about an axis normal to the plane of motion, a single subscript for I is enough to denote the inertia axis. So if a flat lamina has plane motion in xy -plane, then moment of inertia of lamina about the z -axis through \circ is denoted by I_o .

For three dimensional motion, where the components of rotation may occur about more than one axis, we use double ^{sub}scripts to preserve notational symmetry with product of inertia terms. Thus, the moments of inertia about the x - y and z -axes are denoted by

$$I_{xx}, I_{yy}, I_{zz}$$

Physical Significance of Moment of Inertia



Consider a body of mass rotating about an axis OO' with an (accelerati) angular acceleration α . All particles of the body move in parallel planes

which are normal to the rotation axis OO' . Any one of these planes may be considered the plane of motion of body although usually the plane containing the centre of mass is taken as the plane of motion. An element of mass dm has a component of acceleration tangent to its circular path equal to $r\alpha$ ($a = r\alpha$).

By Newton's 2nd law of motion the resultant tangential force on this element dm is

$$r\alpha dm.$$

The moment of this force about the axis OO' is

$$r(r\alpha dm) = r^2\alpha dm.$$

and the sum of the moments of these forces for all elements is

$$\int r^2\alpha dm$$

For a rigid body α is same for all radial lines (all radial lines move with same angular velocity, acc in a given time) and may

18

be taken out of the integral sign.
The remaining integral

$\int r^2 dm$
is known as the moment of inertia
I of the mass m about the axis OO
and is

$I = \int r^2 dm$
This integral represents an important of
a body and is involved in the force analysis
of any body that has rotational acceleration
about a given axis.

Now inertia is intrinsic ()
reluctance or resistance (to accept) which a body
offers to accept a change in its state of motion.

In case linear motion the mass of a body
is the measure of reluctance to the translational
acceleration. In case of rotational motion the
moment of inertia is a measure of resistance
to rotational acceleration of the body.

The moment of inertia of a body is a
function of

- (i) the mass of body (ii) the distribution of the
mass of body (e.g. size and shape)
- (iii) and the position of axis of rotation.

Radius of Gyration

It is the distance
k from the axis of rotation such that if
all the mass m of body were concentrated
at distance k, the correct moment of inertia
would be $k^2 m = I$.

The radius of gyration of a mass m about an

axis for which the moment of inertia is I is

$$k = \sqrt{\frac{I}{m}}$$

Thus k is the measure of distribution of mass of a given body about the axis in question.

Centre of Mass of System of Particles

Sum of Mass Moments about C.M.

Centre of mass of a system of particles m_1, m_2, \dots, m_n with p.v. \underline{r}_i relative to O is defined as

$$\underline{r}_{c.m.} = \frac{\sum m_i \underline{r}_i}{\sum m_i = m}$$

Co-ordinates of c.m. are given by

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i} \quad \bar{y} = \frac{\sum m_i y_i}{\sum m_i}$$

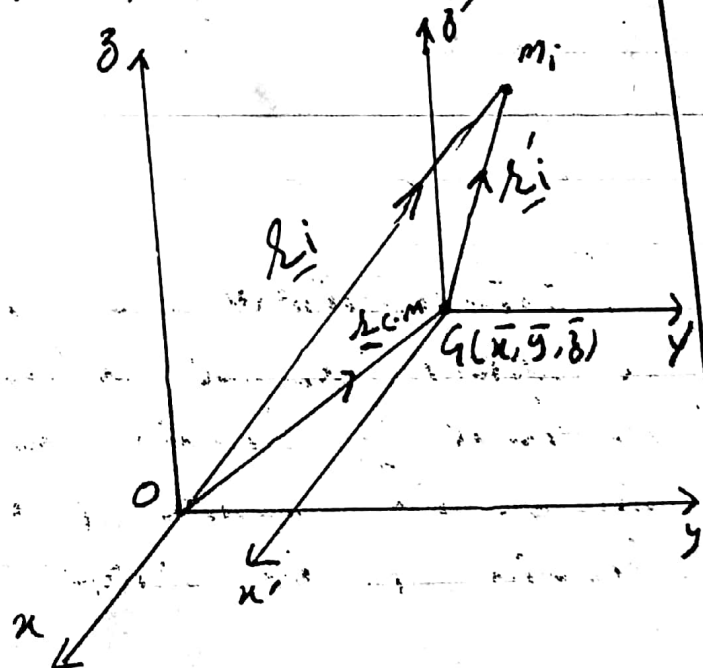
$$\bar{z} = \frac{\sum m_i z_i}{\sum m_i}$$

where $\sum m_i \underline{r}_i$ is called sum of mass moments of all the particles.

Let $G(x', y', z')$ be co-ordinate system with origin at the c.m. Let

\underline{r}_i be p.v. w.r.t O and \underline{r}'_i be p.v. w.r.t G of i th particle m_i .

$$\underline{r}_i = \underline{r}_{c.m.} + \underline{r}'_i$$



$$\underline{x}_i' = \underline{x}_i - \underline{x}_{c.m.}$$

$$\begin{aligned} \Rightarrow \sum m_i \underline{x}_i' &= \sum m_i \underline{x}_i - \underline{x}_{c.m.} \sum m_i \\ &= \sum m_i \underline{x}_i - \left(\frac{\sum m_i \underline{x}_i}{\sum m_i} \right) \sum m_i \\ &= \sum m_i \underline{x}_i - \sum m_i \underline{x}_i \end{aligned}$$

$$\sum m_i \underline{x}_i' = 0 \quad \rightarrow \textcircled{1}$$

\Rightarrow Sum of mass moments (about) w.r.t C.m is zero. So we can define C.m of a system as a point w.r.t which the sum of linear moments is zero.

From $\textcircled{1}$

$$\sum m_i x_i' = 0$$

$$\sum m_i y_i' = 0$$

$$\sum m_i z_i' = 0$$

We note that x_i' is distance of particles from $y_i'z_i'$ -plane (or from y' -axis, or z' -axis)

Parallel Axis Theorem

OR

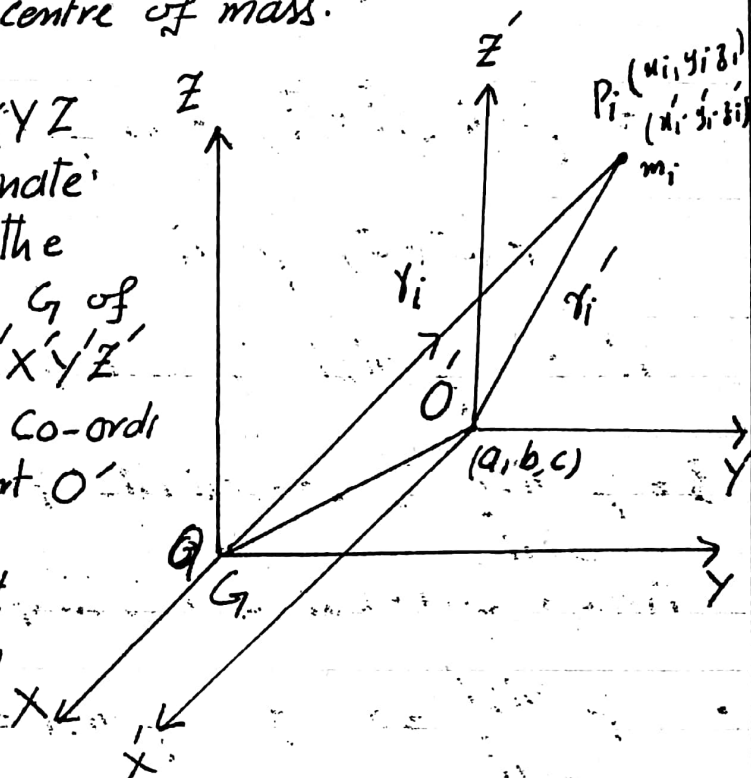
Principle of Parallel Axes # (Transfer of Axis)

Statement# (1) The moment of inertia of a system about a line is equal to moment of inertia about a parallel line through the centre of mass plus the moment of inertia of the whole mass concentrated at

the centre of mass about line l

(2) The product of inertia of a system about two lines l, m (not passing through the c.m.) is equal to the product of inertia about parallel lines (one parallel to l , other parallel to m) plus product of inertia about l, m of the whole mass $M = \sum m_i$ placed at the centre of mass.

Proof# Let $OXYZ$ be the Co-ordinate system through the centre of mass G of a system. Let $O'X'Y'Z'$ be a parallel Co-ordinates through point O' with Co-ordinates (a, b, c) w.r.t Co-ordinate system through G .



Let m_i placed at point P_i has Co-ordinates (x_i, y_i, z_i) and (x'_i, y'_i, z'_i)

Then $x'_i = x_i - a$

$y'_i = y_i - b$

$z'_i = z_i - c$

Now

$$I_{x'x'} = \sum_i m_i (y_i^2 + z_i^2)$$

$$= \sum_i m_i [(y_i - b)^2 + (z_i - c)^2]$$

$$= \sum_i m_i [y_i^2 - 2by_i + \overset{22}{b^2} + z_i^2 - 2cz_i + c^2]$$

$$= \sum_i m_i y_i^2 - 2b \sum_i m_i y_i + \sum_i m_i z_i^2 + (b^2 + c^2) \sum_i m_i - 2c \sum_i m_i z_i$$

$$= \sum_i m_i (y_i^2 + z_i^2) - 2b \sum_i m_i y_i - 2c \sum_i m_i z_i + (b^2 + c^2) \sum_i m_i$$

Now $\sum_i m_i y_i$ is sum of moments of

$\frac{\sum_i m_i y_i}{\sum m_i}$ is co-ordinate of C.M along y-axis

through C.M. But C.M is itself origin. Therefore

$$\frac{\sum_i m_i y_i}{\sum m_i} = 0 \Rightarrow \sum_i m_i y_i = 0$$

$$\text{Similarly } \sum m_i z_i = 0$$

Hence

$$I_{x'x'} = \sum_i m_i (y_i^2 + z_i^2) + (b^2 + c^2) M$$

$$= I_{xx} + (b^2 + c^2) M$$

$$\text{Here } M = \sum_i m_i$$

I_{xx} = Moment of inertia about x-axis through G

$I_{x'x'}$ = Moment of inertia about a line parallel to x-axis

$(b^2 + c^2)M$ = Moment of inertia of total mass

²³
~~about~~ at c.m about line x' -axis.
Similarly

$$I_{y'y'} = I_{yy} + M(a^2 + c^2)$$

$$I_{z'z'} = I_{zz} + M(a^2 + b^2)$$

Product of inertia about x' -axis, y' -axis

$$= I_{x'y'} = \sum_i m_i x'_i y'_i$$

$$= \sum_i m_i (x_i - a)(y_i - b)$$

$$= \sum_i m_i x_i y_i - a \sum_i m_i y_i - b \sum_i m_i x_i + \sum_i m_i ab$$

$$= \sum_i m_i x_i y_i + 0 + 0 + \sum_i m_i ab$$

$$= I_{xy} + Mab$$

\Rightarrow Product of inertia about two lines x' , y'

= Product of inertia about parallel lines, through c.m

+ Product of inertia of whole mass M at c.m about lines x' , y'

Hence the parallel axes theorem is proved.

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Perpendicular Axes Theorem

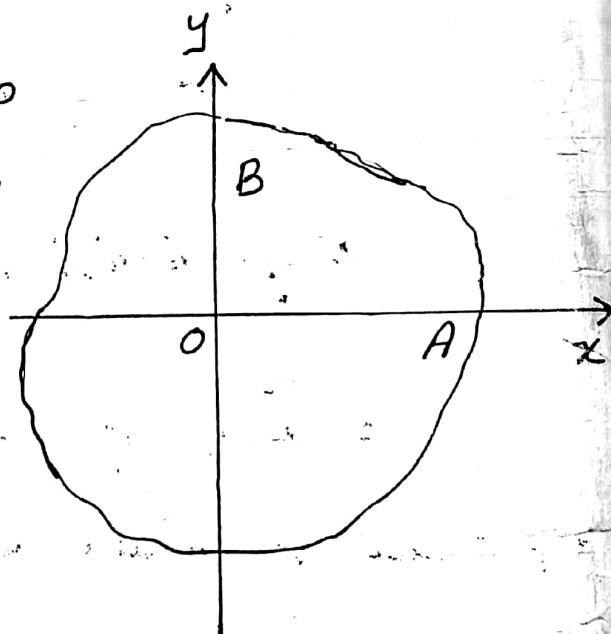
OR

Lamina Theorem

Statement # If A and B are the moments of inertia of a (flat) lamina about two perpendicular axes in its plane, then its moment of inertia C about the line through the point of intersection of these axes and perpendicular to these is given by

$$C = A + B$$

Proof # Taking the two perpendicular lines in lamina as x -axis and y -axis respectively and the line through their point of intersection perpendicular to the lamina as z -axis, we have



$$A = \int (y^2 + z^2) dm = \text{Moment of inertia about } x\text{-axis}$$

$$= \int y^2 dm \quad \text{as } z = 0$$

$$B = \int (z^2 + x^2) dm = \text{Moment of inertia about } y\text{-axis}$$

$$= \int x^2 dm \quad z = 0$$

Moment of inertia about z-axis

$$= C = \int (x^2 + y^2) dm$$

$$= \int x^2 dm + \int y^2 dm$$

$$C = A + B \quad \text{proved} \rightarrow \textcircled{1}$$

If k_1, k_2, k_3 are radii of gyration about x-axis, y-axis and z-axis respectively, then

$$A = Mk_1^2 \quad B = Mk_2^2 \quad C = Mk_3^2$$

Then from $\textcircled{1}$ we have

$$Mk_3^2 = Mk_1^2 + Mk_2^2$$

$$\Rightarrow k_3^2 = k_1^2 + k_2^2$$

Remarks# 1) # Lamina theorem is useful to calculate moment of inertia of any plane (lamina, ring, triangular framework etc) about certain axes.

2) # This theorem cannot be applied to three dimensional rigid bodies.

3) # The three axes in theorem must be mutually perpendicular and concurrent, although none of them need pass through the centre of mass of body.

4) # The parallel axes theorem given before is more general and can be applied to three dimensional rigid bodies.

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Routh's Rule

The following rule given by Dr. Routh is useful for remembering the moments of inertia of certain symmetrical solid bodies about their principal axes and symmetry axes.

M.I about an axis of symmetry

$$= \text{Mass} \times \left[\frac{\text{The sum of squares of perp. semi-axes}}{3 \text{ or } 4 \text{ or } 5} \right]$$

↓
Value of k^2 (square of
radius of gyration)

The denominator is 3 or 4 or 5 according as the body is rectangular (including the rod and a square) or elliptical (including a circle) or ellipsoidal (including a sphere)

Moment of Inertia of a Uniform Rod

Question # (a) # Find the moment of inertia of a thin uniform rod about axis (i) through the centre (C.M) and perp. to rod (ii) through the centre and inclined at angle θ to rod (iii) through one end of rod and perp to rod.

(b) # Find the moment of inertia of a uniform rod about an axis (i) through the centre (ii) through centre and inclined at angle θ to the rod (iii) through one end of rod and perp to the rod.

Sol # (a) Thin Uniform Rod #

Let AB^{2a} be a thin rod of uniform

22

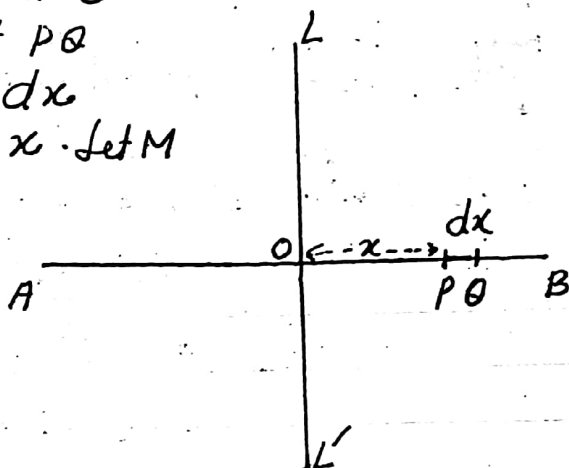
density ρ so that mass per unit length is same through out the rod.

(i) Suppose we require the moment of inertia of the rod about the axis LOL' which bisects the rod perpendicularly at O .

Consider an element PQ of differential length dx at a distance $OP = x$. Let M be total mass of the rod.

Differential mass

element $= dm = \rho dx$



But $\rho = \frac{\text{Mass}}{\text{Length}} = \frac{M}{2a}$

$$\Rightarrow dm = \left(\frac{M}{2a}\right) dx$$

Moment of inertia of mass dm about LOL' is

$$dI_{LL'} = dI_{yy} = x^2 dm$$

Moment of inertia of rod is

$$= I_{LL'} = I_{yy} = \int_{-a}^a x^2 dm$$

$$= \int_{-a}^a \left(\frac{M}{2a}\right) \cdot x^2 dx$$

$$= \frac{M}{2a} \left[\frac{x^3}{3} \right]_{-a}^a = \frac{M}{2a} \left[\frac{a^3}{3} + \frac{a^3}{3} \right]$$

$$I_{yy} = \frac{1}{3} Ma^2 \Rightarrow \text{Result}$$

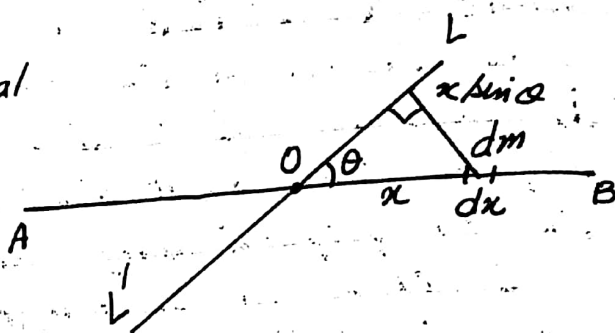
(ii) # Suppose we want the moment of inertia about an axis LOH' through mid point and

as shown:

Consider a differential length element dx at distance x from O .

Perp distance of dx

from $LOL' = x \sin \theta$



Differential Mass = $dm = \rho dx$

Differential moment of inertia of dm about LOL'

$$= dI_{LL'} = x^2 \sin^2 \theta dm$$

$$= \rho \sin^2 \theta x^2 dx$$

Moment of inertia of rod about LOL' is

$$I_{LL'} = \int_{-a}^a \rho \sin^2 \theta x^2 dx$$

$$= \rho \sin^2 \theta \left[\frac{x^3}{3} \right]_{-a}^a$$

$$= \frac{M}{2a} \sin^2 \theta \left[\frac{a^3}{3} + \frac{a^3}{3} \right]$$

$$= \frac{M}{2a} \sin^2 \theta \cdot \frac{2a^3}{3}$$

$$= M \frac{a^2}{3} \sin^2 \theta$$

$I_{LL'} = M \frac{a^2}{3} \sin^2 \theta$

→ Result

Deductions: Moment of inertia of thin uniform rod about an axis through centre and inclined at angles of 30° , 45° , 60° , 90° is given

$$\frac{Ma^2}{12}, \quad \frac{Ma^2}{6}, \quad \frac{Ma^2}{4}, \quad \frac{Ma^2}{3}$$

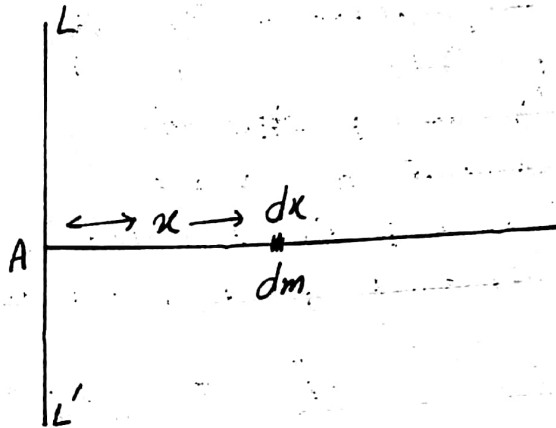
We note that moment of inertia about an axis which has greater inclination to rod is greater.

(iii) # Moment of inertia about an axis passing through end A is given by

$$I_{LL'} = \int_0^{2a} x^2 dm$$

$$= \int_0^{2a} x^2 \cdot \frac{M}{2a} dx$$

$$= \frac{M}{2a} \left[\frac{x^3}{3} \right]_0^{2a} = \frac{4}{3} Ma^2$$



OR By parallel axis theorem

Moment of inertia about an axis through one end \$L\$ to rod

= Moment about centroidal \$L'\$ axis

+ Moment of whole mass at centre about axis through end

$$= \frac{1}{3} Ma^2 + Ma^2 = \frac{4}{3} Ma^2$$

(b)

Uniform Rod # ^{AB}

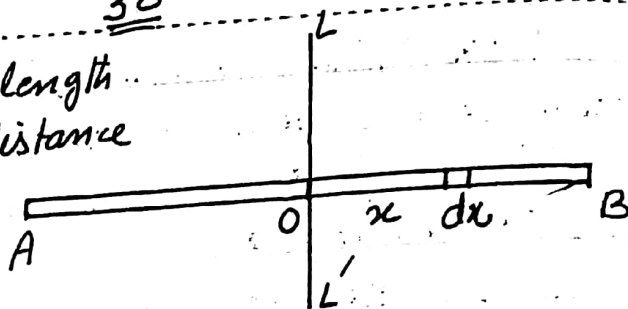
Consider a uniform rod of length \$2a\$ and of cross-sectional area \$\alpha\$. \$O\$ is middle point of rod.

Uniform density \$\rho\$ of rod is

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{2a\alpha}$$

$$\left[\begin{array}{l} \text{Volume} \\ = \text{Area of cross-section} \times \text{length} \end{array} \right]$$

- (i). Consider differential length element dx at distance x from the centre of rod.



Volume of differential element = αdx

Mass of the element = $\rho \alpha dx = dm$

Moment of mass dm about $L-L'$ is

$$dI_{LL'} = x^2 \rho \alpha dx$$

Moment of inertia of rod is

$$I_{LL'} = \rho \alpha \int_{-a}^{+a} x^2 dx$$

$$= \left(\frac{M}{2\alpha} \right) \alpha \left[x^3 \right]_{-a}^a$$

$$= \frac{M}{2\alpha} \left[\frac{2a^3}{3} \right]$$

$$\boxed{I_{LL'} = \frac{Ma^2}{3}} \longrightarrow \text{Result}$$

- (ii) Moment of inertia about centroidal axis inclined at an angle θ is given by

$$dm = \rho \alpha dx$$

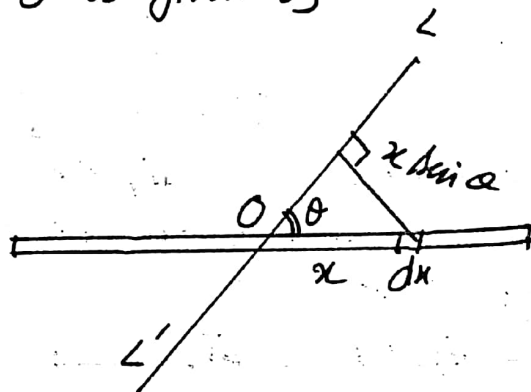
$$dI_{LL'} = x^2 \sin^2 \theta dm$$

$$= \rho \alpha \sin^2 \theta \cdot x^2 dx$$

$$I_{LL'} = \int_{-a}^a \rho \alpha \sin^2 \theta \cdot x^2 dx$$

$$= \left(\frac{M}{2\alpha} \right) \alpha \cdot \sin^2 \theta \int_{-a}^a x^2 dx$$

$$I_{LL'} = \frac{Ma^2}{3} \sin^2 \theta$$



(iii) Moment of inertia about an axis LL' passing through end A is given by

$$dm = \rho dx$$

$$dI_{LL'} = x^2 dm$$

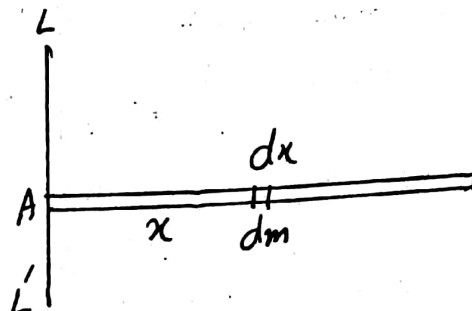
$$= (\rho dx) x^2$$

$$= \frac{M}{2a} \cdot \rho \cdot x^2 \cdot dx$$

$$= \frac{M}{2a} x^2 dx$$

$$I_{LL'} = \frac{M}{2a} \int_0^a x^2 dx = \frac{M}{2a} \cdot \frac{8a^3}{3} = \frac{4Ma^2}{3}$$

$$I_{LL'} = \frac{4Ma^2}{3} \longrightarrow \text{Result}$$



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Moment of Inertia of Uniform Thin Rod About an axis Parallel to Rod

Question# Find the moment of inertia of a uniform thin rod of Mass M and length $2a$ about an axis parallel to the rod at distance d from it.

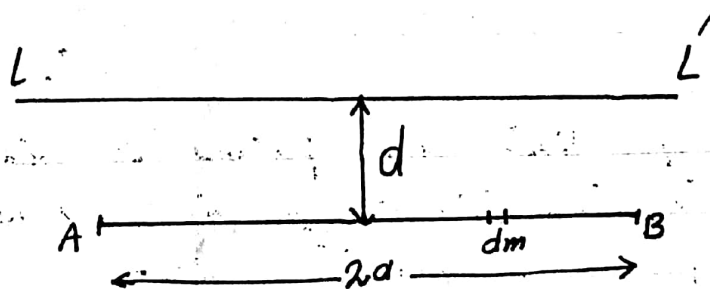
Sol#

Consider rod

$AB = 2a$ of uniform density ρ . Consider

a differential mass element dm .

Moment of dm about $LL' = dI_{LL'} = d^2 dm$



Moment of inertia of rod about LL' is

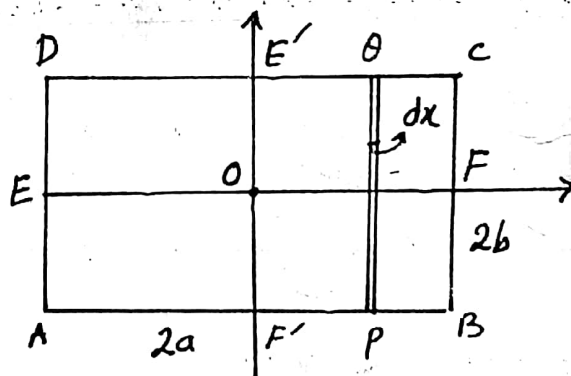
$$I_{LL'} = \int d^2 dm = d^2 \int dm$$

$$= d^2 M \quad \int dm = M$$

Moment of Inertia of Uniform Rectangular Lamina

Question # Find the moment of Inertia of a uniform rectangular lamina of sides $2a, 2b$ about
 (i) Centroidal axes parallel to sides
 (ii) an axis passing through centre and perp to the lamina
 (iii) Product of inertia about centroidal axes parallel to sides.

Sol # Let ABCD be a uniform rectangular lamina of density ρ and mass M . If E, F are mid points of sides AD, BC and E', F' are mid points of sides CD, AB, then $EF, E'F'$ are centroidal axes because lamina is uniform.



Moment About EF

Consider a thin strip PQ of width dx parallel to side AD. Thin strip is like thin rod of uniform density and axis EF passes through its centre.

Moment of inertia of strip is

$$dI_{xx} = dI_{EF} = \frac{1}{3} b^2 dm$$

(This moment is calculated by formula of moment of inertia of a thin uniform rod about an axis passing through middle point of the rod and in the plane of rod)
 is Moment of inertia of the whole Lamina about EF

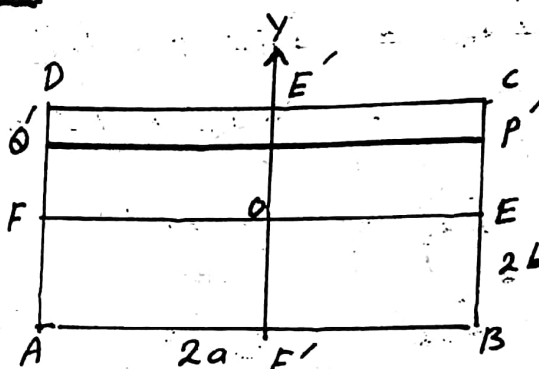
$$I_{EF} = I_{xx} = \frac{1}{3} b^2 \int dm$$

$$I_{xx} = \frac{1}{3} b^2 M \quad \rightarrow \textcircled{1} \quad \because \int dm = M \text{ whole mass of Lamina}$$

(i) # Moment About E'F'

Consider a thin strip P'O' of width dy parallel to side AB.

Then P'O' is a thin uniform rod of length $2a$ and E'F' is an axis at middle point of it. so



Moment of inertia of P'O' about E'F' is

$$dI_{E'F'} = \frac{1}{3} dy^2 dm \quad (\text{By thin uniform rod method})$$

\Rightarrow Moment of the whole Lamina about E'F' is

$$I_{yy} = I_{E'F'} = \frac{1}{3} a^2 \int dm$$

$$I_{yy} = \frac{1}{3} a^2 M \quad \rightarrow \textcircled{2}$$

(ii) # Moment of Inertia about an axis through centre and perp. to Lamina

Now by Theorem of perpendicular axes the moment of inertia about an axis through centre and perpendicular to lamina is

$$\frac{1}{3} b^2 M + \frac{1}{3} a^2 M = \frac{1}{3} (a^2 + b^2) M$$

Moment About sides of lamina

By using parallel axes theorem we can find moment of inertia about sides as.

Moment of inertia about side AB
 = Moment of inertia about side CD
 = Moment about EF + Moment of whole mass M at O about AB or CD

$$I_{AB} = I_{EF} + Mb^2$$

$$= \frac{1}{3}Mb^2 + Mb^2 = \frac{4}{3}Mb^2$$

$$I_{AB} = \frac{4}{3}Mb^2 \Rightarrow \text{Result}$$

$$I_{CD} = \frac{4}{3}Mb^2$$

Similarly

$$I_{AD} = I_{BC} = I_{E'F'} + Ma^2$$

$$= \frac{1}{3}Ma^2 + Ma^2$$

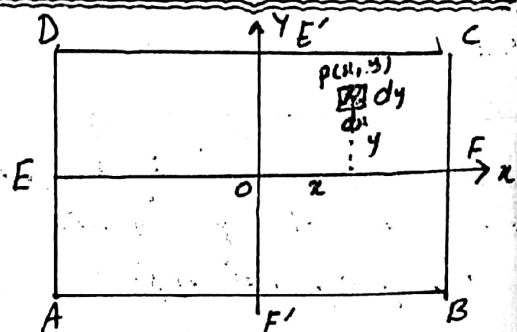
$$I_{AD} = I_{BC} = \frac{4}{3}Ma^2 \Rightarrow \text{Result}$$

(iii) # Product of Moment of Inertia About Centroidal axes

Consider a differential area element $dx dy$ at point $P(x, y)$.

Mass of this element is

$$dm = \rho dx dy$$



Product of inertia of mass element dm about x -axis & y -axis is

$$dI_{xy} = xy dm$$

$$= \rho xy dx dy$$

So product of inertia of the whole lamina about x -axis and y -axis is

$$I_{xy} = \int_{-b}^b \int_{-a}^a \rho xy dx dy$$

$$= \rho \int_{-b}^b \left(\int_{-a}^a x dx \right) y dy$$

$$= \rho \int_{-b}^b \left[\frac{x^2}{2} \right]_{-a}^a y dy$$

$$= \rho \int_{-b}^b \left(\frac{a^2}{2} - \frac{a^2}{2} \right) y dy$$

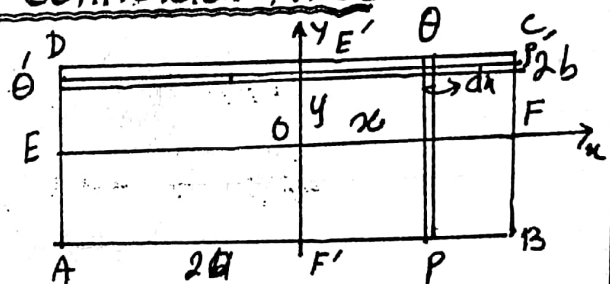
$$= \rho \int_{-b}^b 0 y dy = 0$$

\Rightarrow Centroidal axes are in fact principal axes for lamina.

II. Method

(i) Moment About Centroidal Axes

For moment of inertia about $E'F'$
Consider a strip of width dx parallel to



Side AD.

Length of the strip = $2b$

Area of the strip = $2b dx$

If ρ is density of Lamina, then

mass of the strip = $dm = \rho \cdot 2b dx$

Moment of inertia of dm about EF' is

$$dI_{EF'} = 2b\rho x^2 dx$$

$$\Rightarrow I_{EF'} = 2b\rho \int_{-a}^a x^2 dx$$

$$= 2b\rho \left[\frac{x^3}{3} \right]_{-a}^a$$

$$= 2b\rho \left[\frac{2a^3}{3} \right]$$

$$= \frac{4\rho ba^3}{3}$$

Area of Lamina = $2a \times 2b = 4ab$

total Mass = M

$$\rho = \frac{\text{Mass}}{\text{area}} = \frac{M}{4ab}$$

$$\Rightarrow I_{EF'} = \frac{4ba^3}{3} \cdot \frac{M}{4ab} = \frac{1}{3} Ma^2$$

Similarly considering a strip of width dy parallel to side AB at distance y from O. we have

Length of the strip = $2a$

Area of strip = $2a dy$

$dm = 2a\rho dy$

$$dI_{EF} = y^2 \cdot 2a\rho dy$$

$$I_{EF} = 2a\rho \int_{-b}^b y^2 dy = 2a \cdot \frac{M}{4ab} \cdot \frac{2b^3}{3} = \frac{1}{3} Mb^2$$

OR

Consider a small area element $dx dy$ at point (x, y) of lamina. Then

$$dm = \rho dx dy$$

$$I_{xx} = \iint dI_{xx} = \iint dI_{EF} = \iint y^2 dm$$

$$= \rho \int_{-a-b}^a \int y^2 dy dx$$

$$= \frac{M}{4ab} \int_{-a}^a \left[\frac{y^3}{3} \right]_{-b}^b dx$$

$$= \frac{M}{4ab} \int_{-a}^a \frac{2b^2}{3} \cdot dx$$

$$= \frac{M}{4ab} \cdot \frac{2b^2}{3} \int_{-a}^a dx$$

$$= \frac{M}{4ab} \cdot \frac{2b^2}{3} \cdot 2a = \frac{1}{3} M b^2$$

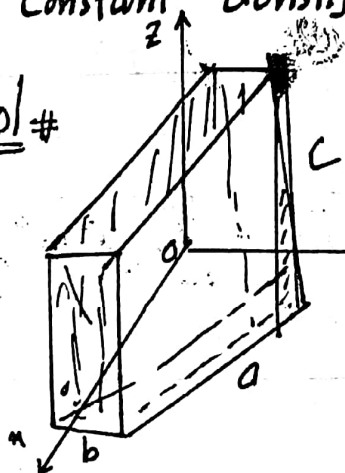
Similarly $I_{yy} = I_{E'F'} = \frac{1}{3} M a^2$

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Uniform Rectangular Block

Question# Find I_x, I_y, I_z for the rectangular solid of constant density ρ as shown

Sol#



The origin lies at the centre of block
Here $dm = \rho dx dy dz$

$$I_x = \iiint (y^2 + z^2) \rho dx dy dz$$

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} (y^2 + z^2) \rho \, dx \, dy \, dz$$

Since the integrand is an even function.
therefore

$$I_x = 8 \int_0^{c/2} \int_0^{b/2} \int_0^{a/2} (y^2 + z^2) \rho \, dx \, dy \, dz$$

$$= 8\rho \int_0^{c/2} \int_0^{b/2} (y^2 + z^2) \Big|_x^0 dy \, dz$$

$$= 8\rho \cdot \frac{a}{2} \int_0^{c/2} \left[\frac{y^3}{3} + z^2 y \right]_{y=0}^{y=b/2} dz$$

$$= 4\rho a \int_0^{c/2} \left(\frac{b^3}{24} + \frac{z^2 b}{2} \right) dz$$

$$= 4\rho a \left(\frac{b^3 c}{48} + \frac{c^3 b}{48} \right) = \frac{4abc\rho}{12} (b^2 + c^2)$$

Now Volume of block = abc

Total Mass = M

$$\rho = \frac{M}{abc}$$

$$\Rightarrow I_x = \frac{4abc}{12} \cdot \frac{M}{abc} (b^2 + c^2)$$

$$= \frac{M}{12} (b^2 + c^2)$$

Similarly

$$I_y = \frac{M}{12} (a^2 + c^2) \quad \& \quad I_z = \frac{M}{12} (a^2 + b^2)$$

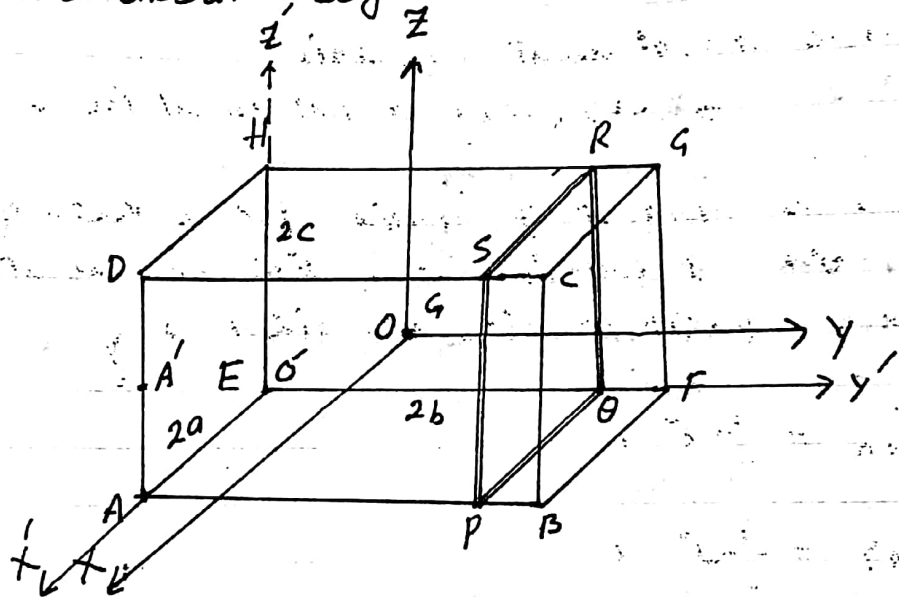
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M.I of Homogeneous Rectangular

Parallelepiped

Question # Calculate the moment of inertia of homogeneous (uniform) rectangular parallelepiped about centroidal axes, about edges, about axes through centre of faces and perp to centroidal axes. Also find the product of inertia about centroidal axes.

Sol # and about edges.



Consider a parallelepiped of edges $2a$, $2b$, $2c$ with centroid (C.M) G . Consider axes through G parallel to respective edges. Let ρ be the uniform density of the parallelepiped.

About Centroidal Axes

To find the moment of inertia about centroidal y -axis, consider a small rectangular element parallel to face $ADHE$ and perpendicular to y -axis. Let dy be its very small thickness as compared the whole.

dimension of parallelepiped.

Dimensions of rectangular element PQRS are $2a, 2c$

$$\text{Area of element} = 2a \times 2c = 4ac$$

$$\text{Now } \rho dy = \text{Mass per unit volume} \times \text{thickness}$$

$$= \text{Mass per unit area} \rightarrow \textcircled{1}$$

When thickness t of flat plate is very small, then mass moment and area moment are related as

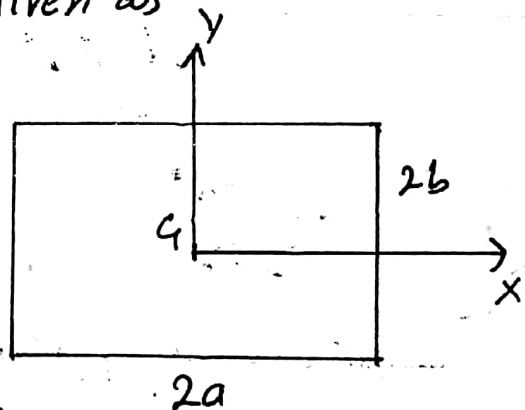
Mass moment about any axis

$$= (\rho t) (\text{Area moment about the same axis})$$

Mass moment of inertia for a rectangular lamina of length $2a$ & width $2b$ about centroidal axes parallel to sides are given as

$$I_{Gx} = \frac{1}{3} a^2 M$$

$$I_{Gy} = \frac{1}{3} b^2 M$$



and Area moments are

$$I_{Gx} = \frac{1}{3} a^2 A$$

where A is total area

$$I_{Gy} = \frac{1}{3} b^2 A$$

where A is total area

Polar Moment about an axis through G perpendicular to lamina is

$$I_{Gx} + I_{Gy}$$

Moment of rectangular strip about y -axis
 $= dI_{yy} = \text{Sum of moments about centroidal}$

41

axes parallel to sides

$$\begin{aligned} dI_{yy} &= \left(\frac{1}{3}a^2A\right) \rho dy + \left(\frac{1}{3}c^2A\right) \rho dy \\ &= \frac{1}{3} \rho (a^2 + c^2) dy \cdot A \\ &= \frac{1}{3} \frac{M}{8abc} (a^2 + c^2) dy \cdot 4ac \\ &= \frac{M}{6b} (a^2 + c^2) dy \end{aligned}$$

OR

Volume of rectangular strip $= 4ac dy$

Mass of the strip $= dm = \rho 4ac dy$

$$\begin{aligned} &= \frac{M}{8abc} \cdot 4ac dy \\ &= \frac{M}{2b} dy \end{aligned}$$

Polar Moment of strip about axis (y-axis) through its centre and perp to its plane is

$$\begin{aligned} dI_{yy} &= \frac{1}{3} dm (a^2 + c^2) \\ &= \frac{1}{3} \cdot \frac{M}{2b} (a^2 + c^2) dy \\ &= \frac{M}{6b} (a^2 + c^2) dy \end{aligned}$$

Moment of parallelepiped about y-axis

$$I_{yy} = \int dI_{yy} = \frac{M}{6b} \int_{-b}^b (a^2 + c^2) dy$$

$$I_{yy} = \frac{M}{6b} (a^2 + c^2) \left| y \right|_b^{-b}$$

$$= \frac{M}{6b} (a^2 + c^2) \cdot 2b$$

$$I_{yy} = \frac{1}{3} M (a^2 + c^2)$$

Similarly

$$I_{xx} = \frac{1}{3} M (b^2 + c^2)$$

$$I_{zz} = \frac{1}{3} M (a^2 + b^2)$$

About Edges

About Edge # AD

Taking G, origin

Co-ordinates point A are (a, -b, 0)

AD is parallel to z-axis and distance d between them is

$$d = AG = \sqrt{a^2 + b^2}$$

Moment of Inertia about AD by parallel axis theorem is

$$I_{AD} = I_{zz} + \text{Moment of the whole mass } M \text{ at } G \text{ about AD}$$

$$= \frac{1}{3} M (a^2 + b^2) + M d^2$$

$$= \frac{1}{3} M (a^2 + b^2) + M (a^2 + b^2)$$

$$= \frac{4}{3} M (a^2 + b^2)$$

Similarly about edge EH is

$$I_{EH} = \frac{4}{3} M (a^2 + b^2) \quad \text{By symmetry}$$

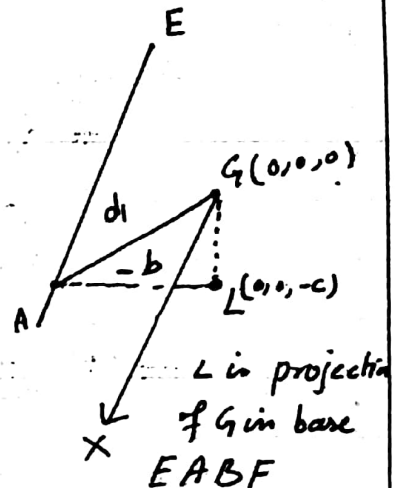
About Edge EA #

EA is parallel to X-axis and its distance d_1 from X-axis is calculated as

$$d_1 = \sqrt{b^2 + c^2}$$

Moment About EA is

$$\begin{aligned} I_{EA} &= I_{xx} + M d_1^2 \\ &= \frac{1}{3} M (b^2 + c^2) + M (b^2 + c^2) \\ &= \frac{4}{3} (b^2 + c^2) M \end{aligned}$$



Similarly we can calculate moments about other edges

$$I_{EF} = \frac{4}{3} (a^2 + c^2) M$$

By Routh's Rule #

Since the centroidal axes through G are symmetry axes, therefore we can apply Routh's Rule to memorize the results as

$$I_{xx} = \text{Mass} \times \frac{(\text{Sum of squares of per-semi axes})}{3}$$

$$\text{Similarly} = \frac{1}{3} M (b^2 + c^2)$$

$$I_{yy} = \frac{1}{3} M (a^2 + c^2)$$

$$I_{zz} = \frac{1}{3} M (a^2 + b^2)$$

Ind Method (Direct Method)

$$I_{xx} = \int_V \rho (y^2 + z^2) dV$$

$$= \rho \int_{-c}^c \int_{-b}^b \int_{-a}^a (y^2 + z^2) dx dy dz \quad \left[\because \rho \text{ is Constant} \right]$$

$$= \rho \cdot |x| \int_{-c}^c \int_{-b}^b (y^2 + z^2) dy dz$$

$$= \rho (2a) \int_{-c}^c \left[\frac{y^3}{3} + z^2 y \right]_{-b}^b dz$$

$$= 2a \rho \int_{-c}^c \left(\frac{b^3}{3} + z^2 b + \frac{b^3}{3} + z^2 b \right) dz$$

$$= 2a \rho \int_{-c}^c \left(\frac{2b^3}{3} + 2bz^2 \right) dz$$

$$= 2a \rho \left[\frac{2b^3}{3} z + \frac{2b}{3} z^3 \right]_{-c}^c$$

$$= 2a \rho \left[\frac{2b^3}{3} c + \frac{2b}{3} c^3 + \frac{2b^3}{3} c + \frac{2b}{3} c^3 \right]$$

$$= 2a \rho \left[\frac{4b^3 c}{3} + \frac{4b}{3} c^3 \right]$$

$$= 2a \rho \cdot \frac{4}{3} bc (b^2 + c^2)$$

$$= \rho \frac{8abc}{3} (b^2 + c^2)$$

45
 putting $\rho = \frac{M}{8abc}$

$$I_{xx} = \frac{M}{8abc} \cdot \frac{8abc}{3} (b^2 + c^2)$$

$$= \frac{1}{3} M (b^2 + c^2)$$

Similarly

$$I_{yy} = \frac{1}{3} M (a^2 + c^2)$$

$$I_{zz} = \frac{1}{3} M (a^2 + b^2)$$

About Edges#

Considering O as origin
 we can find moment about edges.

$$I_{EA} = I_{O'X'} = \int_V \rho (y^2 + z^2) dV$$

$$= \rho \int_0^{2c} \int_0^{2b} \int_0^{2a} (y^2 + z^2) dx dy dz$$

$$= \rho \int_0^{2c} \int_0^{2b} (y^2 + z^2) dy dz$$

$$= 2\rho a \int_0^{2c} \int_0^{2b} (y^2 + z^2) dy dz$$

$$= 2\rho a \int_0^{2c} \left[\frac{y^3}{3} + z^2 y \right]_0^{2b} dz$$

$$= 2\rho a \int_0^{2c} \left(\frac{8b^3}{3} + 2z^2 b \right) dz$$

$$= 2\rho a \left[\frac{8b^3}{3} z + \frac{2z^3}{3} b \right]_0^{2c}$$

46

$$= \rho a \cdot 2 \left| \frac{b^3 c}{3} + \frac{c^3 b}{3} \right|$$

$$= \rho \frac{2abc}{3} (b^2 + c^2) \cdot 16$$

$$= \frac{32}{3} \rho abc (b^2 + c^2)$$

$$\text{but } \rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{2a \cdot 2b \cdot 2c} = \frac{M}{8abc}$$

$$I_{xx'} = \frac{32}{3} \cdot \frac{M}{8abc} \cdot abc (b^2 + c^2)$$

$$= \frac{4}{3} M (b^2 + c^2)$$

Similarly

$$I_{yy'} = \frac{4}{3} M (a^2 + c^2)$$

and

$$I_{zz'} = \frac{4}{3} M (a^2 + b^2)$$

Product of Inertia

(a) About Edges #

Considering O as origin and axes along edges products of inertia about edges are calculated as

$$I_{xy'} = \int \rho xy dv$$

$$\text{Here } dv = dx dy dz$$

$$= \rho \int_0^{2a} x dx \int_0^{2b} y dy \cdot \int_0^{2c} dz$$

$$\begin{aligned}
 &= \rho \int_0^{2a} \left| \frac{x^2}{2} \right|_0^{2a} \cdot \int_0^{2b} \left| \frac{y^2}{2} \right|_0^{2b} \cdot \int_0^{2c} |z|_0^{2c} \\
 &= \rho \left(\frac{4a^2}{2} \right) \cdot \left(\frac{4b^2}{2} \right) \cdot 2c \\
 &= \rho \cdot \frac{32a^2b^2c}{4} \\
 &= \rho \cdot 8a^2b^2c \\
 &= \frac{M}{8abc} \cdot 8a^2b^2c = M_{ab}
 \end{aligned}$$

Similarly $I_{x'z'} = M_{ac}$

$I_{y'z'} = M_{bc}$

About Centroidal Axes

$$\begin{aligned}
 I_{xy} &= \rho \int_V xy \, dv \\
 &= \rho \int_{-a}^a \int_{-b}^b \int_{-c}^c xy \, dz \, dy \, dx
 \end{aligned}$$

\therefore the integrand is an even function in z but is an odd function in x & y . So

$$I_{xy} = \rho \int_{-c}^c 0 \, dz = 0$$

Similarly

$$I_{yz} = I_{zx} = 0$$

Deduction for Uniform Cube of Edge $2a$

Moments of inertia for parallelepiped about centroidal axes are

$$I_{xx} = \frac{1}{3} M (b^2 + c^2)$$

$$I_{yy} = \frac{1}{3} M (a^2 + c^2)$$

$$I_{zz} = \frac{1}{3} M (a^2 + b^2)$$

Moments of inertia for cube about centroidal axes are

$$I_{xx} = \frac{1}{3} M (a^2 + a^2) = \frac{2}{3} M a^2$$

$$I_{yy} = \frac{1}{3} M (a^2 + a^2) = \frac{2}{3} M a^2$$

$$I_{zz} = \frac{1}{3} M (a^2 + a^2) = \frac{2}{3} M a^2$$

Moment of inertia for parallelepiped about edge AD is

$$I_{AD} = \frac{1}{3} M (a^2 + b^2)$$

For cube

$$I_{AD} = \frac{1}{3} M (a^2 + a^2)$$

$$= \frac{2}{3} M a^2$$

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M-I of Uniform Elliptic Lamina or Disc

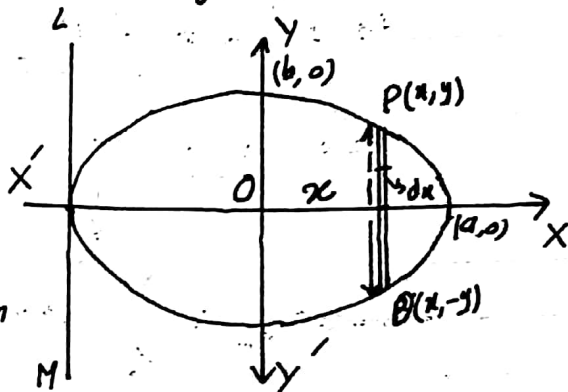
Question # Find the moment of Inertia of a uniform elliptic Lamina of major and minor axes $2a$, $2b$ about

- (i) Centroidal axes
- (ii) an axis through intersection of centroidal axes and perp to the plane of Lamina
- (iii) tangents at the ends of centroidal axes
- (iv) Also deduce all the above results for a uniform circular lamina of radius a
- (v) Find moment of

inertia of circular lamina about an axis through any point on its circumference and perpendicular to its plane

Sol/# Let O be c.m of elliptic Lamin and symmetry axes through be taken as axes.

Dividing disc into strips parallel to YY' , consider a such strip PQ at distance x from O and of width. Let ρ be uniform density.



Equation of the boundary of lamina is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y = \frac{b}{a} \sqrt{a^2 - x^2}$$

Area of the thin strip $PQ = 2y dx$
differential mass of the strip is

$$dm = \rho \cdot 2y dx$$

x -axis passes through the centre of this thin rod of length $2y$.

Moment of inertia of rod PQ about x -axis is

$$dI_{xx} = \frac{1}{3} dm \cdot y^2$$

$$= \frac{1}{3} \cdot 2\rho y \cdot y^2 dx \quad \because dm = 2\rho y dx$$

$$= \frac{2}{3} \rho y^3 dx$$

Moment of inertia of disc about x -axis is

$$I_{xx} = \int_{-a}^a \frac{2}{3} \rho y^3 dx$$

50

$$\text{Let } (x, y) = (a \cos \theta, b \sin \theta)$$

$$\Rightarrow y = b \sin \theta$$

$$\& x = a \cos \theta$$

$$dx = -a \sin \theta d\theta$$

$$\text{When } x = -a \quad \cos \theta = -1$$

$$\Rightarrow \theta = \pi$$

$$\text{When } x = a \quad \cos \theta = 1$$

$$\Rightarrow \theta = 0$$

$$I_{xx} = -\frac{2}{3} \rho \int_0^{\pi} (b \sin \theta)^3 \cdot (a \sin \theta) d\theta$$

$$= \frac{2}{3} \rho b^3 a \cdot 2 \cdot \int_0^{\pi/2} \sin^4 \theta d\theta$$

By Wallis formula

$$= \frac{4}{3} \rho b^3 a \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a b^3 \rho}{4}$$

$$\text{Area of elliptic Lamina} = \pi ab$$

$$\rho = \frac{\text{Mass}}{\text{Area}} = \frac{M}{\pi ab}$$

$$I_{xx} = \frac{\pi a b^3}{4} \cdot \frac{M}{\pi ab}$$

$$= \frac{M b^2}{4}$$

Similarly

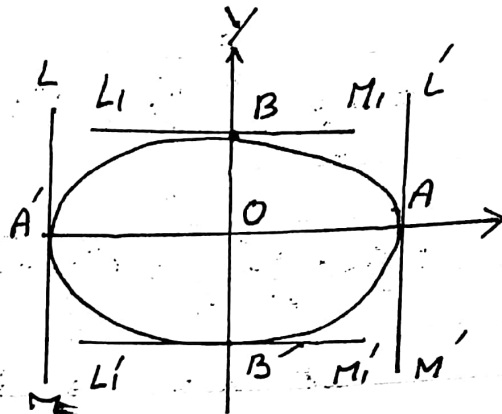
$$I_{yy} = \frac{M a^2}{4}$$

(ii) By theorem of perpendicular axis moment of inertia about an axis

$$\frac{1}{4} M a^2 + \frac{1}{4} M b^2 = \frac{1}{4} M (a^2 + b^2)$$

(iii) M.I About
Tangent at the
ends of Centroidal
axes.

Consider tangents
 $LM, L'M'$ at ends
of major axis. Then
tangents are \perp to
major axis and hence are parallel to
minor axis.



By parallel axes theorem

$$I_{LM} = I_{L'M'} = I_{yy} + Ma^2$$

$$= \frac{1}{4} M b^2 + Ma^2 = \frac{5}{4} M a^2$$

Similarly tangents $L1M1$ & $L1'M1'$ at the ends
of minor axis are parallel to major axis
and

$$I_{L1M1} = I_{L1'M1'} = I_{xx} + Mb^2$$

$$= \frac{1}{4} M b^2 + Mb^2 = \frac{5}{4} M b^2$$

Alternate Method

Here $dm = \rho dx dy$

$$\begin{aligned} I_{xx} &= \int \int_{\text{whole lamina}} y^2 dm = \rho \int \int_{\text{lamina}} y^2 dx dy \\ &= \rho \int_{-a}^a \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 dx dy \end{aligned}$$

$$= 4\rho \int_0^a dx \left| \frac{y^3}{3} \right|_0^{\frac{b}{a}\sqrt{a^2-x^2}}$$

$$= \frac{4\rho}{3} \int_0^a \frac{b^3}{a^3} (a^2-x^2)^{\frac{3}{2}} dx$$

$$= \frac{4\rho}{3} \cdot \frac{b^3}{a^3} \int_0^a (a^2-x^2)^{\frac{3}{2}} dx$$

$$\text{let } x = a \sin \theta$$

$$dx = a \cos \theta d\theta$$

$$\text{When } x=0 \quad \sin \theta = 0 \Rightarrow \theta = 0$$

$$x=a \quad \sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$I_{xx} = \frac{4\rho}{3} \cdot \frac{b^3}{a^3} \int_0^{\frac{\pi}{2}} (a^2 - a^2 \sin^2 \theta)^{\frac{3}{2}} \cdot a \cos \theta d\theta$$

$$= \frac{4\rho}{3} \cdot \frac{b^3}{a^3} \cdot a^4 \int_0^{\frac{\pi}{2}} \cos^4 \theta d\theta$$

$$= \frac{4\rho}{3} b^3 a \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} \quad \text{By Wallis formula}$$

$$= \rho \frac{b^3 a \pi}{4} = \frac{M}{\pi ab} \cdot \frac{b^3 a \pi}{4}$$

$$= \frac{1}{4} M b^2$$

Similarly

$$I_{yy} = \frac{1}{4} M a^2$$

(iv) Deductions for Circular Lamina #

For elliptic Lamina

$$I_{xx} = \frac{1}{4} M b^2 \rightarrow \textcircled{1}$$

$$I_{yy} = \frac{1}{4} M a^2 \rightarrow \textcircled{2}$$

*

For circular lamina of radius (a),
 $b = a$ & we have

$$I_{xx} = \frac{1}{4} Ma^2$$

$$I_{yy} = \frac{1}{4} Ma^2$$

For polar Moment

$$\begin{aligned} I_{zz} &= I_{xx} + I_{yy} \\ &= \frac{1}{4} Ma^2 + \frac{1}{4} Ma^2 \\ &= \frac{1}{2} Ma^2 \end{aligned}$$

For tangents LM, L'M' at ends of diameter
 along x-axis

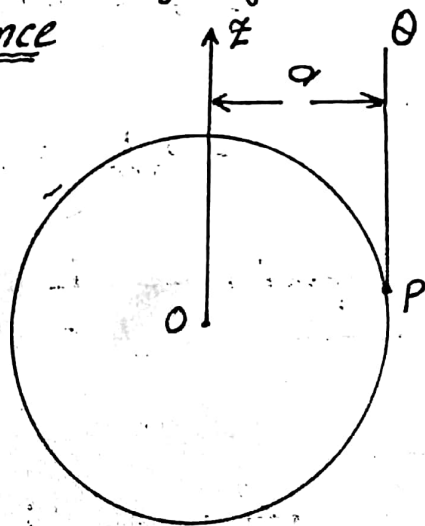
$$I_{LM} = \frac{1}{4} Ma^2 + Ma^2 = \frac{5}{4} Ma^2 = I_{L'M'}$$

About an axis perpendicular to Lamina at any

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Point of Circumference

Any axis PO perpendicular
 to a point P of circumference
 is parallel to central polar
 axis OZ and at distance (a),
 from it. Therefore by parallel
 axis theorem, we have



$$\begin{aligned} I_{PO} &= I_{zz} + Ma^2 \\ &= \frac{1}{2} Ma^2 + Ma^2 \\ &= \frac{3}{2} Ma^2 \quad (\text{proved}) \end{aligned}$$

Triangular Laminas

Question #1 An isosceles triangle Lamina ABC. Find moment of inertia of ABC through the vertex A perpendicular to (i) the opposite side (ii) the plane of the Lamina.

Sol # Since Triangular Lamina is isosceles - Let $AB = AC$ and A be the vertex of Lamina. If we drop perpendicular AD on BC, it will necessarily bisect the vertex and side BC. Let ρ be the uniform density of the Lamina.

taking AD x-axis and y-axis perpendicular to it.

Area of Lamina = $\frac{1}{2}ah$.

$$\text{Density of Lamina} = \frac{M}{\frac{1}{2}ah} = \frac{M}{\frac{1}{2} \cdot \frac{a^2}{2} \cot\left(\frac{A}{2}\right)}$$

$$\tan\left(\frac{A}{2}\right) = \frac{\frac{a}{2}}{AD} \Rightarrow AD = \frac{a}{2} \cot\left(\frac{A}{2}\right)$$

Equation of line AC is

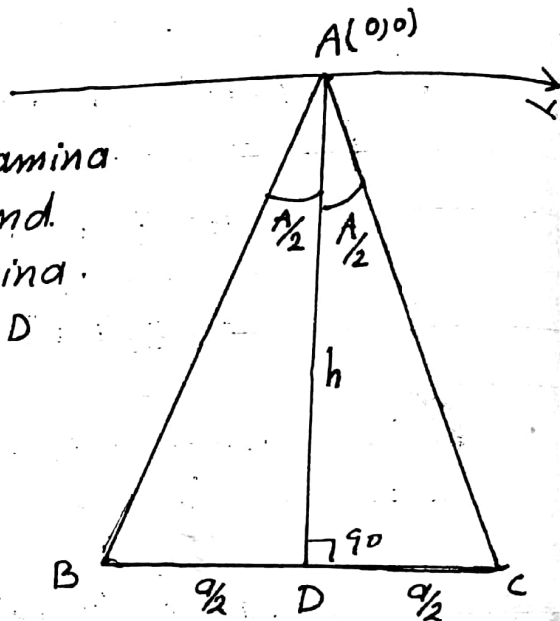
$$y - 0 = \tan\left(\frac{A}{2}\right)(x - 0)$$

$$y = x \tan\left(\frac{A}{2}\right)$$

Equation of line AB is

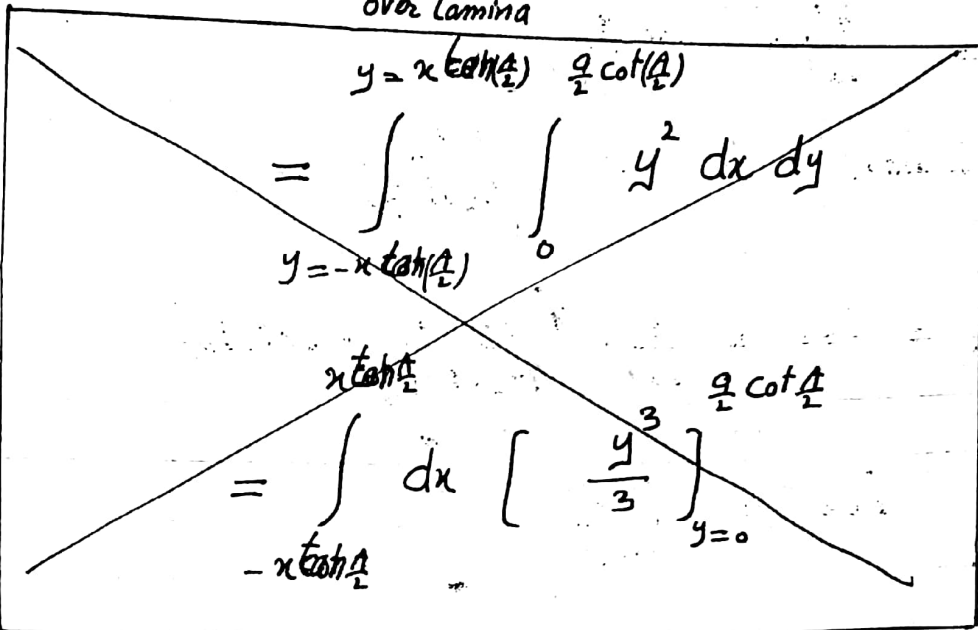
$$y - 0 = \tan\left(2\pi - \frac{A}{2}\right)(x - 0)$$

$$y = -x \tan\left(\frac{A}{2}\right)$$



Moment of Inertia about AD is

$$I_{AD} = I_{xx} = \int \int_{\text{over lamina}} y^2 \rho \, dx \, dy$$



$$= \int_{y = -x \tan(\frac{A}{2})}^{y = x \tan(\frac{A}{2})} \int_0^{\frac{a}{2} \cot(\frac{A}{2})} y^2 \, dx \, dy$$

$$= \int_{-x \tan \frac{A}{2}}^{x \tan \frac{A}{2}} dx \left[\frac{y^3}{3} \right]_{y=0}^{\frac{a}{2} \cot \frac{A}{2}}$$

$$= \rho \int_0^{\frac{a}{2} \cot \frac{A}{2}} \int_{y = -x \tan \frac{A}{2}}^{y = x \tan \frac{A}{2}} y^2 \, dy \, dx$$

$$= 2\rho \int_0^{\frac{a}{2} \cot(\frac{A}{2})} \int_0^{x \tan(\frac{A}{2})} y^2 \, dy \, dx$$

$$= 2\rho \int_0^{\frac{a}{2} \cot(\frac{A}{2})} \left[\frac{y^3}{3} \right]_0^{x \tan(\frac{A}{2})} dx$$

$$= \frac{2\rho}{3} \int_0^{\frac{a}{2} \cot(\frac{A}{2})} x^3 \tan^3(\frac{A}{2}) \, dx$$

$$= \frac{2\rho}{3} \tan^3(\frac{A}{2}) \left[\frac{x^4}{4} \right]_0^{\frac{a}{2} \cot(\frac{A}{2})}$$

$$= \frac{2\rho}{3} \tan^3\left(\frac{A}{2}\right) \cdot \frac{1}{4} \cdot \frac{a^4}{16} \cot^4\left(\frac{A}{2}\right)$$

$$= \frac{a^4}{96} \rho \cot\left(\frac{A}{2}\right)$$

putting $\rho = \frac{M}{\frac{a^2}{4} \cot\left(\frac{A}{2}\right)}$

$$I_{AD} = I_{xx} = \frac{a^4}{96} \cdot \frac{4 \cdot M}{a^2 \cot\left(\frac{A}{2}\right)} \cdot \cot\left(\frac{A}{2}\right)$$

$$I_{xx} = \frac{M a^2}{24}$$

Moment of inertia about y-axis

$$I_{yy} = \int \int_{\text{lamina}} \rho x^2 dx dy$$

$$y = x \tan\left(\frac{A}{2}\right) \quad \frac{a}{2} \cot\left(\frac{A}{2}\right)$$

$$= \rho \int_{y = -x \tan\left(\frac{A}{2}\right)} \int_0^{\frac{a}{2} \cot\left(\frac{A}{2}\right)} x^2 dx dy$$

$$= 2\rho \int_0^{\frac{a}{2} \cot\left(\frac{A}{2}\right)} x^2 dx \int_0^{x \tan\left(\frac{A}{2}\right)} dy$$

$$= 2\rho \int_0^{\frac{a}{2} \cot\left(\frac{A}{2}\right)} x^2 \left| y \right|_0^{x \tan\left(\frac{A}{2}\right)} dx$$

$$= 2\rho \int_0^{\frac{a}{2} \cot(A/2)} x^3 \tan(A/2) dx$$

$$= 2\rho \tan(A/2) \left| \frac{x^4}{4} \right|_0^{\frac{a}{2} \cot(A/2)}$$

$$= 2\rho \tan(A/2) \cdot \frac{a^4}{64} \cot^4(A/2)$$

$$\text{putting } \rho = \frac{M}{\frac{a^2}{4} \cot(A/2)}$$

$$I_{yy} = 2 \cdot \frac{M}{\frac{a^2}{4} \cot(A/2)} \cdot \tan(A/2) \cdot \frac{a^4}{64} \cot^4(A/2)$$

$$= \frac{Ma^2}{8} \cot^2(A/2)$$

(ii) Moment of inertia about an axis through A and perpendicular to the plane is

$$I_{Az} = I_{xx} + I_{yy}$$

$$= \frac{Ma^2}{24} + \frac{Ma^2}{8} \cot^2(A/2)$$

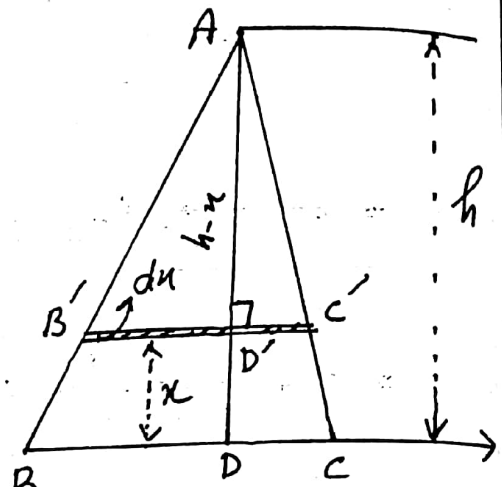
$$= \frac{1}{8} Ma^2 \left(\frac{1}{3} + \cot^2\left(\frac{A}{2}\right) \right)$$

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Question # 2 A uniform triangular lamina ABC. prove that its moment of inertia about BC is $\frac{Mh^2}{6}$, where h is the distance of A from BC

Sol # Let ABC be uniform triangular lamina. h is distance of A from BC . Considering lamina made up of thin strips parallel to BC .

Consider a strip $B'C'$ of width dx at distance x from BC .



Now we know that moment of inertia of rod of mass M about an axis at distance d & parallel to rod is Md^2 .

If dm is mass of the strip, then its moment of inertia about BC is

$$dI = x^2 dm$$

To find dm , we find length of strip:

Consider triangles ABC & $AB'C'$

$\therefore AC, A'C'$ are collinear, AB', AB are collinear

Also $BC, B'C'$ are parallel

\therefore triangles are similar and their sides will be proportional

Also $\triangle ACD$ & $\triangle AC'D'$ are similar

$$\Rightarrow \frac{D'C'}{DC} = \frac{h-x}{h}$$

$$D'C' = \left(\frac{h-x}{h}\right) DC \rightarrow \textcircled{1}$$

Similarly $\triangle ABD$ & $\triangle AB'D'$ are similar

$$\Rightarrow \frac{B'D'}{BD} = \frac{h-x}{h}$$

$$\Rightarrow B'D' = \left(\frac{h-x}{h}\right) BD \rightarrow \textcircled{2}$$

Adding ① & ②

$$D'C' + B'D' = \left(\frac{h-x}{h}\right) (DC + BD)$$

$$B'C' = \frac{h-x}{h} BC$$

$$B'C' = \left(\frac{h-x}{h}\right) a \quad [\because BC = a]$$

Therefore

dm = mass of the strip

$$= \rho B'C' dx$$

$$= a \rho \left(\frac{h-x}{h}\right) dx$$

where ρ is mass per unit area

$$\therefore dI = a \rho \cdot x^2 \left(\frac{h-x}{h}\right) dx$$

$$= \frac{a \rho}{h} (hx^2 - x^3) dx$$

Moment of inertia of the lamina about BC

$$= I = \frac{a \rho}{h} \int_0^h (hx^2 - x^3) dx$$

$$= \frac{a \rho}{h} \left[\frac{hx^3}{3} - \frac{x^4}{4} \right]_0^h$$

$$= \frac{a \rho}{h} \left[\frac{h^4}{3} - \frac{h^4}{4} \right]$$

$$= \frac{a \rho}{h} \frac{h^4}{12} = \frac{a \rho \cdot h^3}{12}$$

$$\text{Area of lamina} = \frac{1}{2} ah$$

$$e = \frac{M}{\frac{1}{2}ah}$$

$$I = \frac{ah^3}{12} \cdot \frac{M}{\frac{1}{2}ah}$$

$$I = \frac{1}{6} M h^2 \quad \longrightarrow \text{Result}$$

Method-II

Let $AO = h$.

taking BC along x-axis
and $OA = h$ as y-axis

from ΔAOC

$$\frac{OC}{h} = \tan(90^\circ - c) = \cot c$$

$$OC = h \cot c$$

\Rightarrow Co-ordinates of

C are $(h \cot c, 0)$

From ΔAOB

$$\frac{OB}{h} = \tan(90^\circ - B) = \cot B$$

$$\Rightarrow OB = h \cot B$$

\Rightarrow Co-ordinates of B are $(-h \cot B, 0)$

Also

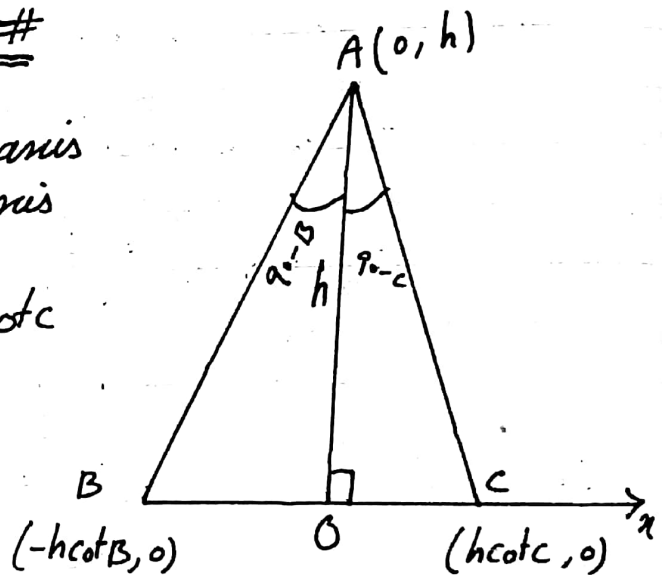
$$BO + OC = h \cot B + h \cot c$$

$$a = h(\cot B + \cot c)$$

$$\frac{a}{h} = \cot B + \cot c \quad \longrightarrow \textcircled{1}$$

Equation of AB is

$$\frac{y-0}{h} = \frac{x+h \cot B}{h \cot B}$$



$$\Rightarrow y = \frac{x + h \cot B}{\cot B}$$

$$\Rightarrow x = y \cot B - h \cot B \rightarrow (2)$$

Equation of AC

$$\frac{y-0}{h-0} = \frac{x-h \cot C}{-h \cot C}$$

$$y = \frac{x - h \cot C}{- \cot C}$$

$$\Rightarrow x = -y \cot C + h \cot C \rightarrow (3)$$

Moment of inertia about AO is

$$I_{AO} = I_{xx} = \rho \int_{x=y \cot B - h \cot B}^{x=-y \cot C + h \cot C} \int_0^h y^2 dy dx$$

$$= \rho \int_0^h y^2 \left[x \right]_{y \cot B - h \cot B}^{-y \cot C + h \cot C} dy$$

$$= \rho \int_0^h y^2 \left[-y \cot C + h \cot C - (y \cot B - h \cot B) \right] dy$$

$$= \rho \int_0^h y^2 \left\{ -y (\cot C + \cot B) + h (\cot B + \cot C) \right\} dy$$

$$= \rho \int_0^h y^2 (h - y) (\cot B + \cot C) dy$$

By ① $\frac{a}{h} = \cot B + \cot C$

$$I_{OA} = \frac{\rho a}{h} \int_0^h (hy^2 - y^3) dy$$

$$= \frac{\rho a}{h} \left[\frac{hy^3}{3} - \frac{y^4}{4} \right]_0^h$$

$$= \frac{\rho a}{h} \left\{ \frac{h^4}{3} - \frac{h^4}{4} \right\} = \frac{\rho a}{h} \left(\frac{h^4}{12} \right)$$

$$= \frac{\rho a h^3}{12}$$

Area of Lamina = $\frac{1}{2} ah$

$$\rho = \frac{M}{\frac{1}{2} ah}$$

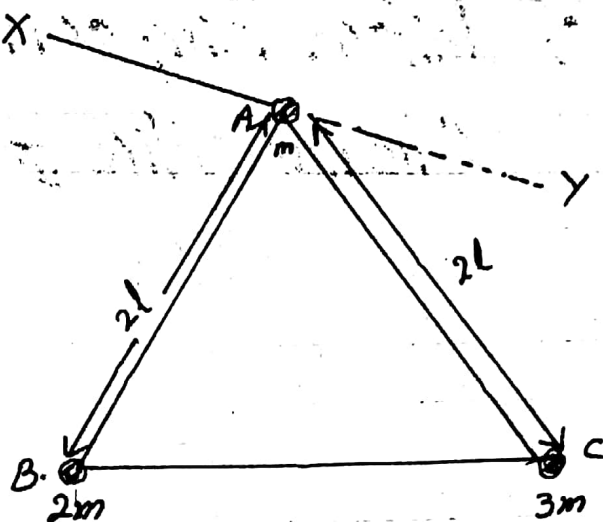
$$I_{OA} = \frac{M}{\frac{1}{2} ah} \cdot \frac{ah^3}{12} = \frac{1}{6} Mh^2$$

⇒ Result

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Question # Three light rods, each of length $2l$, are jointed together to form a triangle. Three particles A, B, C of mass $m, 2m, 3m$ are fixed to vertices of the triangle. Find the moment of inertia of the resulting body about

- an axis through A perpendicular to the plane ABC,
- an axis passing through A and mid point of BC



B is distant $2l$ from the axis XY
 So moment of inertia of B about XY is

$$I_B = 2m(2l)^2$$

Similarly

$$I_C = 3m(2l)^2$$

$$I_A = m(0)^2$$

Therefore moment of inertia of the body about XY is

$$I_{XY} = 2m(2l)^2 + 3m(2l)^2 + m(0)^2$$

$$= 20ml^2$$

(b)

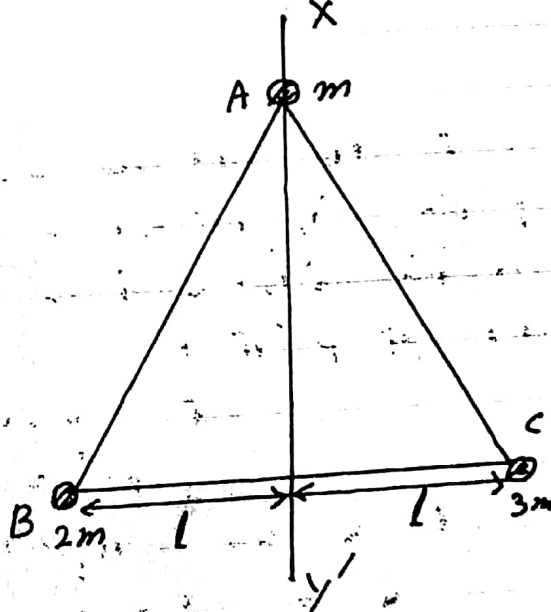
$$I_A \text{ about } X'Y' = m(0)^2$$

$$I_B \text{ about } X'Y' = 2m(l)^2$$

$$I_C \text{ about } X'Y' = 3m(l)^2$$

Therefore moment of inertia of the body about $X'Y'$ is

$$I_{X'Y'} = m(0)^2 + 2m(l)^2 + 3ml^2 = 5ml^2$$



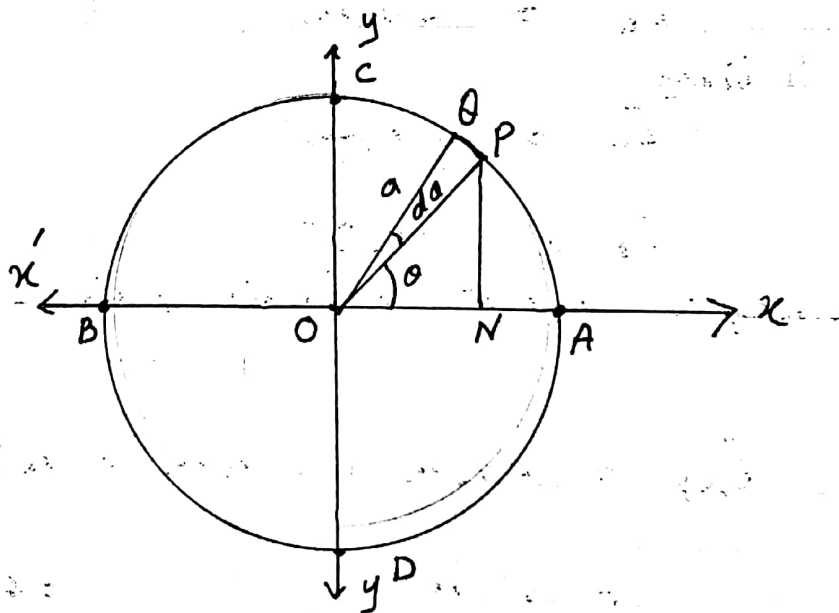
M.I of Uniform Ring or Hoop or Uniform

Circular Wire

Question# Find the moment of Inertia of a uniform ring of mass M and radius a , about

- (i) a diameter of ring
- (ii) a tangent
- (iii) an axis through its centre and perpendicular to the plane of ring
- (iv) an axis at any point of ring and perpendicular to the plane of ring

Sol #



Let AB and CD be two perpendicular diameters of the ring through centre O . Let P be any point on the ring. Consider a small arc element of mass dm at point P .

$$\text{Let } \angle XOP = \theta$$

$$PN = a \sin \theta$$

Let ρ be uniform density of the ring, then

$$\rho = \frac{M}{\text{length}} = \frac{M}{2\pi a}$$

65

Diameter AB is taken along x-axis.
 $db = \overline{PQ} = a d\theta$
 $dm = \rho a d\theta$

Moment of inertia of dm about AB i.e. x-axis is

$$dI_{AB} = dI_{xx} = (a^2 \sin^2 \theta) \cdot dm$$

$$= a^2 \sin^2 \theta \cdot \rho a d\theta$$

$$= a^3 \rho \sin^2 \theta d\theta$$

Moment of Inertia of the whole ring about AB is

$$I_{AB} = I_{xx} = a^3 \rho \int_0^{2\pi} \sin^2 \theta d\theta$$

$$= a^3 \rho \int_0^{2\pi} \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^3 \rho}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi}$$

$$= \frac{a^3 \rho}{2} \left[(2\pi - 0) - \frac{1}{2} (\sin 4\pi - \sin 0) \right]$$

$$= \rho a^3 \pi$$

putting $\rho = \frac{M}{2\pi a}$

$$I_{AB} = \frac{M}{2\pi a} a^3 \pi = \frac{1}{2} M a^2$$

$I_{AB} = \frac{1}{2} M a^2$

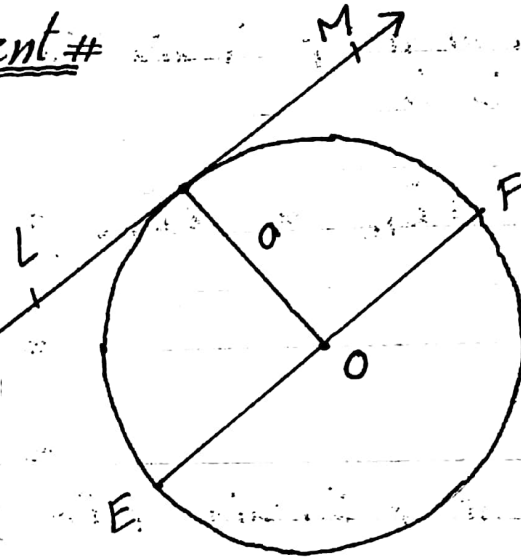
 \longrightarrow Result

By symmetry, we have

$$I_{CD} = I_{yy} = \frac{1}{2} Ma^2$$

(ii) # About tangent #

Let LM be the tangent at any point of the ring and EF be diameter parallel to the tangent. Then



Moment of inertia about diameter is

$$I_{EF} = \frac{1}{2} Ma^2$$

By parallel axes Theorem the moment of inertia about tangent LM is

$$I_{LM} = I_{EF} + Ma^2$$

$$= \frac{1}{2} Ma^2 + Ma^2$$

$$= \frac{3}{2} Ma^2$$

$$I_{\text{tangent}} = \frac{3}{2} Ma^2$$

→ Result.

(iii) About an Axis Through centre & Perp to plane of ring

Let OZ be an axis through

67

Centre O of the ring and perpendicular to the plane of ring. Then

$$I_{Oz} = I_{xx} + I_{yy}$$

$$= \frac{1}{2}Ma^2 + \frac{1}{2}Ma^2$$

$$= Ma^2$$

OR

directly, we have

$$dI_{Oz} = a^2 dm$$

$$= a^2 \rho a d\theta$$

$$= a^3 \rho d\theta$$

$$\Rightarrow I_{Oz} = a^3 \rho \int_0^{2\pi} d\theta = a^3 \rho \left| \theta \right|_0^{2\pi}$$

$$= a^3 \rho (2\pi - 0)$$

$$= 2a^3 \rho \cdot \pi$$

$$= 2a^3 \frac{M}{2\pi a} \cdot \pi = Ma^2$$

$I_{Oz} = Ma^2$

Corollary # Find moment of inertia of semi-circular ring about diameter and about an axis through centre and perpendicular to the plane of ring. Also calculate the same for

for one quarter of the ring and for three quarters of the ring.

Sol # $dm = \rho a d\theta$

$$dI_{AB} = a^3 \rho \sin^2 \theta d\theta$$

Moment of inertia of semi-ring about diameter is

$$I_{AB} = a^3 \rho \int_0^\pi \sin^2 \theta d\theta$$

$$= a^3 \rho \int_0^\pi \left(\frac{1 - \cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^3 \rho}{2} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^\pi$$

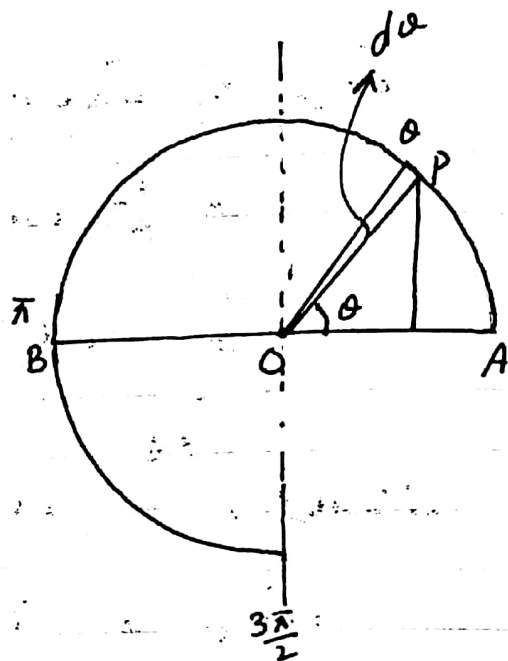
$$= \frac{a^3 \rho}{2} (\pi) = \frac{a^3}{2} \cdot \frac{M}{2\pi a} \cdot \pi$$

$$= \frac{1}{4} Ma^2 = \text{half of the Moment of inertia of ring about diameter}$$

$$I_{AB} = \frac{1}{4} Ma^2$$

Moment of inertia of semi-ring about an axis through centre and perpendicular to the plane of ring is

$$I_{Oz} = \rho a^3 \int_0^\pi d\theta = \rho a^3 \pi = \frac{1}{2} Ma^2$$



= half of the moment of inertia of the ring about the same axis.

For One Quarter of Ring

The moment of inertia of one quarter of ring about diameter is

$$I_{AB} = a^3 \rho \int_0^{\pi/2} \sin^2 \theta d\theta$$

$$= a^3 \frac{M}{2\pi a} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \quad \text{Wallis formula}$$

$$= \frac{1}{8} M a^2$$

= One quarter of the moment of inertia of the ring about diameter

Wallis formula $\int_0^{\pi/2} (\sin^n \theta \text{ or } \cos^n \theta) d\theta$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \cdots \frac{2}{3} \quad \text{when } n \text{ is odd}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \cdots \frac{1}{2} \cdot \frac{\pi}{2} \quad n \text{ is even}$$

The moment of inertia of one quarter of ring about an axis through centre and perpendicular to the plane of ring is

$$I_{OZ} = \rho a^3 \int_0^{\pi/2} d\theta = \rho a^3 \left[\theta \right]_0^{\pi/2}$$

$$= \rho a^3 \cdot \frac{\pi}{2} = \frac{M}{2\pi a} \cdot a^3 \cdot \frac{\pi}{2}$$

$$I_{OZ} = \frac{1}{4} M a^2$$

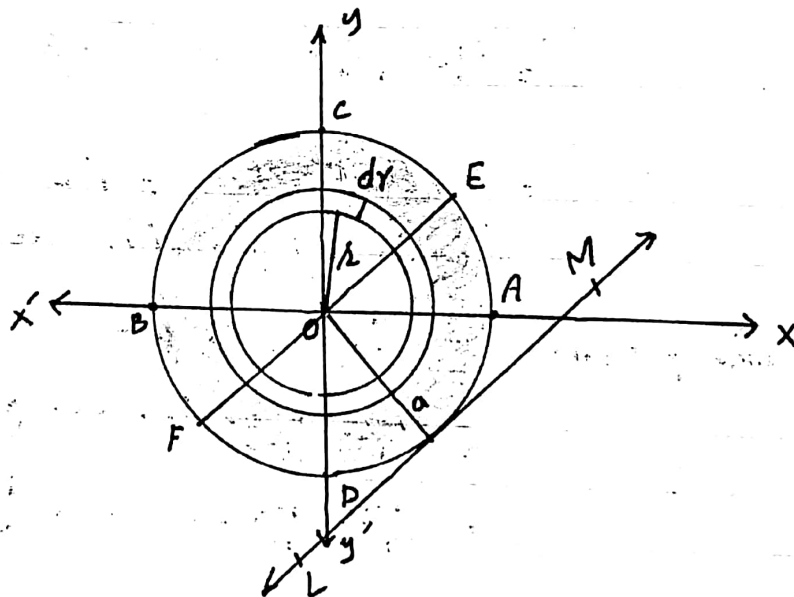
M.I. of Uniform Circular & Semi-Circular

Disc

Question# Find the moment of inertia of a uniform circular disc ^{of radius a} about

- (i) # a diameter of the disc
- (ii) # a tangent
- (iii) an axis through the centre of disc and perpendicular to the plane of disc
- (iv) # an axis through any point of circumference and perpendicular to the plane of disc

Sol #



Let O be the centre of disc and AB along x -axis & CD along y -axis be two perpendicular diameters. Let OZ be 3rd orthogonal axis.

Divide the disc into elements in the form of concentric rings. The thickness of disc is taken negligible as compared to its radius.

Consider a typical thin ring of radius r . If ρ is density of the disc, then the mass of typical ring of radius r is

$$\begin{aligned}
 dm &\approx \rho (\text{Area of ring}) \\
 &= \rho (\text{length} \times \text{breadth}) \\
 &= \rho (2\pi r \times dr) \\
 &= 2\rho\pi r dr
 \end{aligned}$$

The M.I of the ring about AB (diameter) is by previous article

$$\begin{aligned}
 dI_{AB} &= dI_{xx} = \frac{1}{2} r^2 dm \\
 &= \frac{1}{2} r^2 \cdot 2\rho\pi r dr \\
 &= \rho\pi r^3 dr
 \end{aligned}$$

\Rightarrow M.I of the disc about AB (x-axis) is

$$\begin{aligned}
 I_{AB} &= I_{xx} = \rho\pi \int_0^a r^3 dr \\
 &= \rho\pi \left| \frac{r^4}{4} \right|_0^a \\
 &= \frac{\rho\pi}{4} \cdot a^4
 \end{aligned}$$

$$\rho = \frac{M}{\text{area of disc}} = \frac{M}{\pi a^2}$$

$$I_{AB} = \frac{M}{\pi a^2} \cdot \frac{\pi}{4} \cdot a^4$$

$$I_{AB} = \frac{M \cdot a^2}{4}$$

→ Result

Similarly by symmetry, we have

$$I_{CD} = \frac{1}{4} M a^2$$

(ii) By Theorem of parallel axes, the moment of

of inertia about a tangent LM is calculated as
 Let EF be diameter parallel to LM.

Then

$$I_{EF} = \frac{1}{4} Ma^2$$

$$I_{LM} = I_{EF} + Ma^2$$

$$= \frac{1}{4} Ma^2 + Ma^2$$

$$I_{LM} = \frac{5}{4} Ma^2 \longrightarrow \text{Result}$$

(iii) About Centroidal axis Perpendicular to disc #

By perpendicular axes theorem, we have

$$I_{OZ} = I_{xx} + I_{yy}$$

$$= \frac{1}{4} Ma^2 + \frac{1}{4} Ma^2$$

$$I_{OZ} = \frac{1}{2} Ma^2 \longrightarrow \text{Result}$$

(iv) About an axis Through Circumference and Perp. to disc #

Such an axis will be parallel to centroidal perpendicular axis OZ and at distance (a) from it. Therefore by parallel axes theorem moment of inertia about PZ' (PZ' || OZ) is

$$I_{PZ'} = I_{OZ} + Ma^2$$

$$= \frac{1}{2} Ma^2 + Ma^2 = \frac{3}{2} Ma^2$$

Direct Method#

(i) About Diameters#

$$I_{AB} = I_{xx} = \int_{\text{Disc}} y^2 dm$$

$$= \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^a y^2 \rho dy dx$$

$$= \rho \cdot 4 \int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 dx dy$$

$$= 4\rho \int_0^a \left[\frac{y^3}{3} \right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{4}{3}\rho \int_0^a (a^2-x^2)^{3/2} dx$$

Let $x = a \sin \theta$

$dx = a \cos \theta d\theta$

When $x = 0$ $\sin \theta = 0 \Rightarrow \theta = 0$

$x = a$ $\sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$$I_{AB} = I_{xx} = \frac{4\rho}{3} \int_0^{\pi/2} (a^2 - a^2 \sin^2 \theta)^{3/2} \cdot a \cos \theta d\theta$$

$$= \frac{4\rho}{3} \cdot a^4 \int_0^{\pi/2} \cos^4 \theta d\theta$$

$$= \frac{4\rho}{3} \cdot a^4 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = \rho a^4 \frac{\pi}{4}$$

Putting $\rho = \frac{M}{\pi a^2}$

$$I_{AB} = \frac{M}{\pi a^2} \cdot \frac{\pi}{4} a^4$$

$$I_{AB} = \frac{1}{4} M a^2 \longrightarrow \text{Result.}$$

Similarly $I_{yy} = \frac{1}{4} M a^2$

About Centroidal axis perpendicular to Disc #

$$I_{zz} = \rho \iint_{\text{Disc}} (x^2 + y^2) dx dy$$

$$= \rho \int_0^{2\pi} \int_0^a r^2 \cdot r dr d\theta$$

$$= \rho \int_0^{2\pi} \int_0^a r^3 dr d\theta$$

$$= \rho \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^a d\theta$$

$$= \frac{\rho a^4}{4} \int_0^{2\pi} d\theta = \frac{a^4 \rho}{4} 2\pi$$

$$= \frac{a^4 \rho \cdot \pi}{2} = \frac{a^4 \pi}{2} \cdot \frac{M}{\pi a^2}$$

$$I_{zz} = \frac{1}{2} M a^2 \longrightarrow \text{Result.}$$

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Uniform Hollow Right Circular Cylinder

Question # Find the moment of inertia of a uniform right circular hollow cylinder of mass M , radius r and height (length) h about

- (i) # the axis of cylinder
- (ii) # an axis through c.m of cylinder and perp. to axis of the cylinder
- (iii) # an element of the cylinder
- (iv) # a diameter of one end
- (v) # a tangent line which lies in a plane passing through the centre of mass and perpendicular to axis and prove that radius of gyration k about this tangent is given by

$$k^2 = \frac{3r^2}{2} + \frac{h^2}{12}$$

Sol # Divide the cylinder into uniform circular ring all parallel to the ends.

of cylinder or with planes perpendicular to the axis zz' of the cylinder.

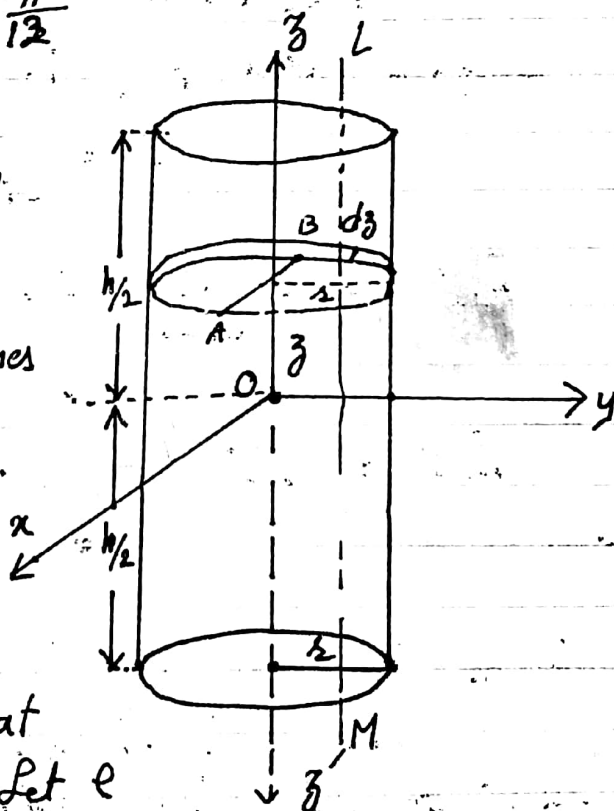
O is centre of mass of the cylinder.

Consider a circular ring of breadth dz at distance z from O . Let ρ be areal density of cylinder.

Area of circular ring = $2\pi r dz$

Mass of the ring is

$$dm = \rho \cdot 2\pi r dz$$



For the ring element axis zz' is centroidal axis perpendicular to its plane.

The moment of inertia of ring about zz' is

$$dI_{zz'} = r^2 dm$$

$$= r^2 \cdot \rho \cdot 2\pi r dz$$

$$= 2\rho\pi r^3 dz$$

$$\Rightarrow I_{zz'} = 2\rho\pi r^3 \int_{-h/2}^{h/2} dz$$

$$= 2\rho\pi r^3 \cdot [z]_{-h/2}^{h/2} = 2\rho\pi r^3 \left[\frac{h}{2} + \frac{h}{2} \right]$$

$$= 2\rho\pi r^3 \cdot h$$

Surface area of cylinder = $2\pi r \cdot h$

$$\rho = \frac{M}{2\pi r h}$$

$$I_{zz'} = 2 \cdot \frac{M}{2\pi r h} \cdot \pi r^3 h = Mr^2$$

$I_{zz'} = Mr^2$

→ Result

(ii) About an axis through c.m & Perp. to axis #

The axis through c.m and perpendicular to the axis of cylinder will be parallel to some diameter of the ring element and diameter of the ring element is centroidal axis for ring. Let Ox be axis through c.m and I_{xx} be the axis of cylinder and AB be

11

diameter of ring parallel to ox .

Moment of inertia of ring about diameter AB is

$$dI_{AB} = \frac{1}{2} r^2 dm$$

$$= \frac{1}{2} r^2 \cdot \rho \cdot 2\pi r dz$$

$$= \rho \pi r^3 dz$$

By parallel axes theorem moment of inertia of ring about ox is

$$dI_{ox} = dI_{AB} + z^2 dm$$

$$= \rho \pi r^3 dz + z^2 \cdot \rho \cdot 2\pi r dz$$

$$I_{ox} = \rho \pi r^3 \int_{-h/2}^{h/2} dz + \rho \cdot 2\pi r \int_{-h/2}^{h/2} z^2 dz$$

$$= \rho \pi r^3 \left[z \right]_{-h/2}^{h/2} + \rho \cdot 2\pi r \left[\frac{z^3}{3} \right]_{-h/2}^{h/2}$$

$$= \rho \pi r^3 \left[\frac{h}{2} + \frac{h}{2} \right] + \rho \cdot \frac{2\pi r}{3} \left[\frac{h^3}{8} + \frac{h^3}{8} \right]$$

$$= \rho \pi r^3 h + \frac{2}{3} \pi r \cdot \rho \cdot \frac{h^3}{4}$$

$$= \rho \pi r h \left(r^2 + \frac{2h^2}{12} \right)$$

$$\text{putting } \rho = \frac{M}{2\pi r \cdot h}$$

$$I_{ox} = \frac{M}{2\pi r h} \cdot \pi r h \left(r^2 + \frac{2h^2}{12} \right)$$

$$I_{ox} = M \left(\frac{r^2}{2} + \frac{h^2}{12} \right)$$

$$\boxed{I_{ox} = M\left(\frac{L^2}{2} + \frac{h^2}{12}\right)} \longrightarrow \text{Result}$$

(iii) About an element of cylinder #

Any line lying on the cylinder and parallel to the axis of the cylinder is called an element of the cylinder.

Let LM be an element of the cylinder. Then LM is parallel to the axis of the cylinder and at distance r from axis. Therefore by parallel axis theorem, we have

$$\begin{aligned} I_{LM} &= I_{gg'} + Mr^2 \\ &= Mr^2 + Mr^2 = 2Mr^2 \end{aligned}$$

$$\boxed{I_{LM} = 2Mr^2} \longrightarrow \text{Result}$$

(iv) About a diameter of one end #

Diameter of one end will be parallel to some centroidal axis perpendicular to the axis of cylinder and at distance $h/2$ from it.

Let (d) be a diameter parallel to the centroidal axis, ox. Then by parallel axes theorem, we have

$$I_d = I_{ox} + M\left(\frac{h}{2}\right)^2$$

$$= M\left(\frac{L^2}{2} + \frac{h^2}{12}\right) + M \cdot \frac{h^2}{4}$$

$$I_d = M\left(\frac{L^2}{2} + \frac{h^2}{12} + \frac{h^2}{4}\right)$$

$$I_d = M\left(\frac{L^2}{2} + \frac{h^2}{3}\right)$$

Moment of inertia about diameter of one end is

$$I_d = M\left(\frac{h^2}{2} + \frac{h^2}{3}\right)$$

→ Result.

(V) About tangent in Centroidal plane \perp to axis #

tangent to the cylinder, which lies in a centroidal plane \perp to axis (or parallel to one end) will be parallel to a centroidal axis perpendicular to the axis of the cylinder.

Let Ox be parallel to tangent line. Then by parallel axis theorem.

$$I_{\text{tangent}} = I_{Ox} + Mk^2$$

$$= M\left(\frac{h^2}{2} + \frac{h^2}{12}\right) + Mk^2$$

$$I_{\text{tangent}} = M\left(\frac{3}{2}h^2 + \frac{h^2}{12}\right)$$

→ Result.

Thus radius of Gyration about tangent is given by

$$k^2 = \frac{3}{2}h^2 + \frac{h^2}{12}$$

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M.I of Uniform Solid Right Circular Cylinder

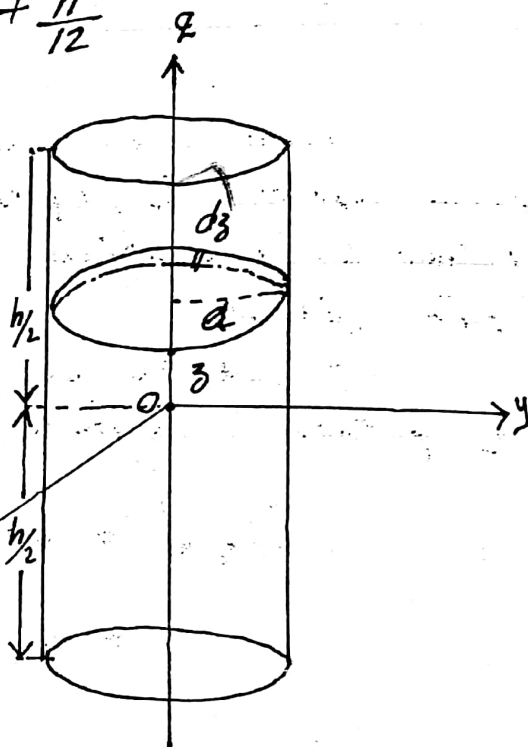
Question# Find the moment of inertia of uniform solid right cylinder of radius a , height h , and mass M about

- (i) # axis of the cylinder
- (ii) # Centroidal axis perpendicular to the axis of cylinder
- (iii) a generator (element) of cylinder

- (iv) # a diameter of one end
 (v) # a tangent line which lies in a plane passing through C.M and perpendicular to axis and prove that the radius of gyration K about this tangent is given by

$$K^2 = \frac{5a^2}{4} + \frac{h^2}{12}$$

Sol# Divide the cylinder into uniform circular discs all concentrated at axis and parallel to the ends of cylinder. Let O be C.M of the cylinder. Consider a circular disc of thickness dz at distance z from O . Let ρ be the density of cylinder.



$$\text{Volume of disc element} = \pi a^2 dz$$

$$\text{Mass of disc element} = \rho \pi a^2 dz = dm$$

z -axis is centroidal axis of disc element and perpendicular to the plane of disc.

So Moment of inertia of disc element about axis of the cylinder is

$$dI_{zz} = \frac{1}{2} dm a^2 \quad (\text{From article of disc})$$

$$= \frac{1}{2} \rho \pi a^4 dz$$

Moment of inertia of the cylinder about axis is

$$I_{zz} = \frac{1}{2} \rho \pi a^4 \int_{-h/2}^{h/2} dz$$

81

$$I_{zz} = \frac{1}{2} \rho \pi a^4 \left[z \right]_{-h/2}^{h/2}$$

$$= \frac{1}{2} \rho \pi a^4 \left[\frac{h}{2} + \frac{h}{2} \right]$$

$$= \frac{1}{2} \rho \pi a^4 h$$

$$\rho = \frac{\text{Mass}}{\text{Volume}} = \frac{M}{\pi a^2 h}$$

$$I_{zz} = \frac{1}{2} \cdot \frac{M}{\pi a^2 h} \cdot \pi a^4 h$$

$$I_{zz} = \frac{1}{2} M k^2 \longrightarrow \text{Result}$$

OR

We can find moment of inertia about axis directly as

$$I_{zz} = \int \rho (x^2 + y^2) dV$$

We use cylindrical polar Co-ordinates.

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$dV = r dr d\theta dz$$

$$0 \leq r \leq a \quad 0 \leq \theta \leq 2\pi \quad -h/2 \leq z \leq h/2$$

$$x^2 + y^2 = r^2 (\cos^2 \theta + \sin^2 \theta) = r^2$$

$$I_{zz} = \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^a r^2 \cdot r dr d\theta dz$$
$$= \rho \int_0^a r^3 dr \int_0^{2\pi} d\theta \int_{-h/2}^{h/2} dz$$

$$= e \frac{a^4}{4} \cdot 2\pi \cdot \left[\frac{h}{2} + \frac{h}{2} \right]$$

$$= \frac{1}{4} e \cdot a^4 \cdot 2\pi \cdot h = \frac{1}{2} e \cdot a^4 \pi h$$

putting $e = \frac{M}{\pi a^2 h}$

$$I_{zz} = \frac{1}{2} M a^2$$

(ii) About Centroidal Axis Perp. to Axis.

Here ox & oy can be taken as centroidal axis (~~or diameteric axis~~) per-to axis

Moment of inertia of disc element about ox (~~diameter~~) can be calculated as.

Let AB be diameter (centroidal axis) of disc element. Then moment of inertia of disc element is

$$\begin{aligned} dI_{AB} &= \frac{1}{4} dm a^2 \\ &= \frac{1}{4} a^2 \cdot e \pi a^2 dz \\ &= \frac{1}{4} a^4 e \pi dz \end{aligned}$$

By parallel axes theorem moment of inertia disc element about ox is

$$dI_{ox} = dI_{AB} + z^2 dm$$

$$= \frac{1}{4} a^4 e \pi dz + e \pi a^2 \cdot z^2 dz$$

Moment of inertia of cylinder about ox is

$$I_{ox} = \frac{1}{4} a^4 e \pi \int_{-h/2}^{h/2} dz + e \pi a^2 \int_{-h/2}^{h/2} z^2 dz$$

$$= \frac{1}{4} a^4 \rho \pi \left| z \right|_{-h/2}^{h/2} + \rho \pi a^2 \left| \frac{z^3}{3} \right|_{-h/2}^{h/2}$$

$$= \frac{1}{4} a^4 \rho \pi \left[\frac{h}{2} + \frac{h}{2} \right] + \rho \pi a^2 \left[\frac{h^3}{24} + \frac{h^3}{24} \right]$$

$$= \frac{1}{4} a^4 \rho \pi \cdot h + \rho \pi a^2 \cdot \frac{h^3}{12}$$

$$= \frac{1}{4} \rho \pi a^2 h \left[a^2 + \frac{h^2}{3} \right]$$

putting $\rho = \frac{M}{\pi a^2 h}$

$$I_{ox} = \frac{1}{4} \cdot \frac{M}{\pi a^2 h} \cdot \pi a^2 h \left[a^2 + \frac{h^2}{3} \right]$$

$$I_{ox} = M \left[\frac{a^2}{4} + \frac{h^2}{12} \right] \rightarrow \text{Result.}$$

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Similarly

$$I_{oy} = M \left[\frac{a^2}{4} + \frac{h^2}{12} \right] \rightarrow \text{Result.}$$

OR

We can calculate moment of inertia about ox & oy directly as

$$I_{ox} = \int \rho (y^2 + z^2) dv$$

using cylindrical polar co-ordinates.

$$x = r \cos \theta \quad y = r \sin \theta \quad z = z$$

$$y^2 + z^2 = r^2 \sin^2 \theta + z^2$$

$$dv = r dr d\theta dz$$

$$I_{ox} = \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^a (r^2 \sin^2 \theta + z^2) r dr d\theta dz$$

$$= \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^a r^3 \sin^2 \theta dr d\theta dz$$

$$+ \rho \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^a r z^2 dr d\theta dz$$

$$= \rho \int_{-h/2}^{h/2} dz \int_0^{2\pi} \sin^2 \theta d\theta \int_0^a r^3 dr$$

$$+ \rho \int_0^a r dr \cdot \int_0^{2\pi} d\theta \int_{-h/2}^{h/2} z^2 dz$$

$$= \rho \left[z \right]_{-h/2}^{h/2} \cdot \left[\frac{r^4}{4} \right]_0^a \int_0^{2\pi} \frac{1 - \cos 2\theta}{2} d\theta$$

$$+ \rho \left[\frac{r^2}{2} \right]_0^a \left[\theta \right]_0^{2\pi} \left[\frac{z^3}{3} \right]_{-h/2}^{h/2}$$

$$= \rho \left[\frac{h}{2} + \frac{h}{2} \right] \frac{a^4}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{2\pi}$$

$$+ \rho \left[\frac{a^2}{2} \right] (2\pi - 0) \cdot \frac{1}{3} \left[\frac{h^3}{8} + \frac{h^3}{8} \right]$$

$$= \rho \cdot h \cdot \frac{a^4}{8} [2\pi] \cdot \rho \cdot \frac{a^2}{2} \cdot 2\pi \cdot \frac{1}{3} \cdot \frac{h^3}{4}$$

$$= \rho \frac{h a^4 \pi}{4} + \frac{\rho a^2 h^3 \pi}{12} = \frac{1}{4} \rho \pi a^2 h \left[a^2 + \frac{h^2}{3} \right]$$

putting $\rho = \frac{M}{\pi a^2 h}$

$$I_{ox} = \frac{1}{4} \frac{M}{\pi a^2 h} \cdot \pi a^2 h \left[a^2 + \frac{h}{3} \right]$$

$$I_{ox} = \frac{1}{4} M \left[a^2 + \frac{h}{3} \right]$$

Similarly

$$I_{oy} = \frac{1}{4} M \left[a^2 + \frac{h}{3} \right]$$

(iv) # About Diameter of one End

Diameter of any flat end of the cylinder will be parallel to some centroidal axis \perp to axis of the cylinder and at distance $h/2$ from it.

Let (d) be diameter of base end parallel to centroidal ox . Then by parallel axes Theorem

$$I_d = I_{ox} + M \left(\frac{h}{2} \right)^2$$

$$= M \left(\frac{a^2}{4} + \frac{h^2}{12} \right) + M \frac{h^2}{4}$$

$$= M \left(\frac{a^2}{4} + \frac{h^2}{12} + \frac{h^2}{4} \right)$$

$$= M \left(\frac{a^2}{4} + \frac{h^2}{3} \right)$$

$$I_d = M \left(\frac{a^2}{4} + \frac{h^2}{3} \right)$$

by M. Hussain Lecturer (Maths)

→ Result.

(v) # About a Generator of Cylinder

Any line on cylinder and parallel to the axis of the cylinder is called generator of the cylinder.

Let LM be a generator of the cylinder. Then LM is parallel to the axis of the cylinder and at distance (a) from the axis. Therefore by parallel axis theorem, we have

$$I_{LM} = I_{zz} + Ma^2$$

$$= \frac{1}{2} Ma^2 + Ma^2$$

$$\boxed{I_{LM} = \frac{3}{2} Ma^2} \longrightarrow \text{Result.}$$

(V) About a tangent in centroidal plane parallel to ends

Tangent to the cylinder, which lies in centroidal plane \perp ar to the axis of cylinder (or parallel to the ends of cylinder) will be parallel to a centroidal axis \perp ar to the axis of the cylinder.

Let ox be parallel to tangent line. then by parallel axis theorem, we have

$$I_{\text{tangent}} = I_{ox} + Ma^2$$

$$= M\left(\frac{a^2}{4} + \frac{h^2}{12}\right) + Ma^2$$

$$\boxed{= M\left(\frac{5a^2}{4} + \frac{h^2}{12}\right)} \longrightarrow \text{Result}$$

Thus radius of Gyration about this tangent is given by

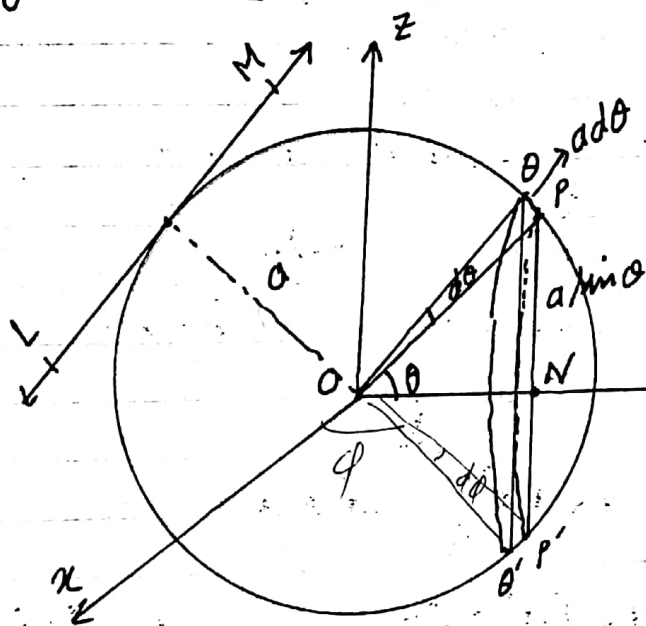
$$k^2 = \frac{5a^2}{4} + \frac{h^2}{12} \quad \text{Proved}$$

Uniform Spherical Shell

Question # Find the moment of inertia of a uniform hollow sphere (spherical shell) of radius a , and Mass of M about

- (i) # a diameter (or symmetry axis)
- (ii) # a tangent at any point of shell.

Sol #



afno dlo
a d f

$$dm \propto a^2 \sin \theta d\theta dx$$

$$I_y = \int_{\text{sphere}} dr^2 (a \sin \theta)^2$$

Let O be the centre of spherical shell. Take axes OX, OY, OZ at O . Divide the sphere into elements by parallel cuts perpendicular to OY . Each element is approximately a circular band (or circular cylinder) and OY is its centroidal axis perpendicular to its diameter. Consider $POP'O'$ such circular band.

Radius of circular slice = $a \sin \theta$

Breadth of " " = $a \cos$

$$\begin{aligned}\text{Area of circular band} &= 2\pi a \sin \theta \cdot a d\theta \\ &= 2\pi a^2 \sin \theta d\theta\end{aligned}$$

Mass of slice = $dm = \rho \cdot 2\pi a^2 \sin\theta d\theta$

The moment of inertia of slice about o_y

$$dI_{yy} = \text{Mass} \times (\text{radius})^2$$

(By method of uniform circular ring about centroidal axis I_{yy} to the plane of ring)

$$dI_{yy} = 2\pi \rho a^2 \sin \theta \, d\theta \cdot a^2 \sin^2 \theta$$

$$= 2\pi \rho a^4 \sin^3 \theta \, d\theta$$

Hence M.I. of hollow sphere about OX, diameter is

$$I_{yy} = \int_0^\pi 2\pi \rho a^4 \sin^3 \theta \, d\theta$$

$$= 2\pi \rho a^4 \int_0^{\pi/2} \sin^3 \theta \, d\theta$$

$$= 4\pi \rho a^4 \int_0^{\pi/2} \sin^3 \theta \, d\theta$$

$$= 4\pi \rho a^4 \cdot \frac{2}{3}$$

By Wallis formula

$$\text{Surface area of sphere} = 4\pi a^2$$

$$\rho = \frac{M}{4\pi a^2}$$

$$I_{yy} = 4 \cdot \frac{M}{4\pi a^2} \cdot a^4 \cdot \frac{2}{3}$$

$$I_{yy} = \frac{2}{3} Ma^2$$

Thus moment of inertia of hollow sphere about diameter is

$$I_{\text{diameter}} = \frac{2}{3} Ma^2$$

→ Result.

(ii) # About Tangent #

tangent at any point of sphere will be parallel to some diameter of the sphere and diameter of sphere is centroidal axis. So we may apply parallel axes theorem.

Let LM be any tangent and d be diameter parallel to LM. By parallel axes theorem, we have

$$I_{LM} = I_d + Ma^2$$

$$= \frac{2}{3} Ma^2 + Ma^2$$

$$= \frac{5}{3} Ma^2$$

So

$$I_{\text{tangent}} = \frac{5}{3} Ma^2$$

→ Result.

Method-II

Let P be a particle of shell of mass m and P have co-ordinates (x, y, z) .

$$\vec{OP} = x\hat{i} + y\hat{j} + z\hat{k}$$

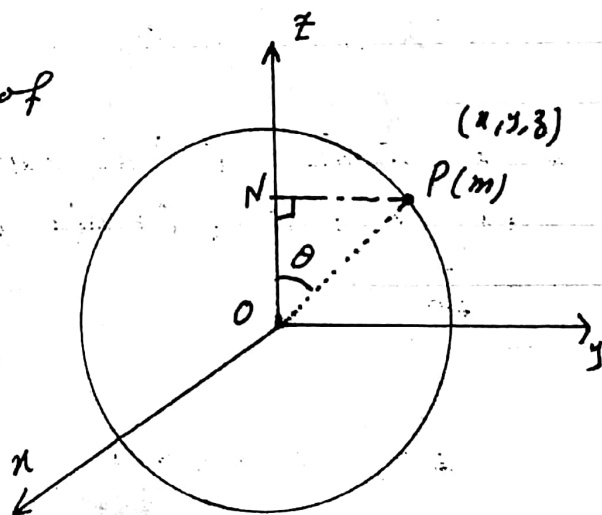
$$PN = |\vec{OP}| \sin \theta$$

$$= |\hat{k} \times \vec{OP}|$$

$$= |\hat{k} \times (x\hat{i} + y\hat{j} + z\hat{k})| = |x\hat{j} - y\hat{i}| = \sqrt{x^2 + y^2}$$

Moment of inertia of m about OZ is

$$mNP^2 = m(x^2 + y^2)$$



So the M.I I of ⁹⁰ shell about OZ is

$$I = \sum m(x^2 + y^2) \rightarrow \textcircled{1}$$

By symmetry moment of inertia about OX, OY is also equal to I . So

$$I = \sum m(y^2 + z^2) \rightarrow \textcircled{2}$$

$$I = \sum m(x^2 + z^2) \rightarrow \textcircled{3}$$

Adding $\textcircled{1} + \textcircled{2} + \textcircled{3}$

$$3I = \sum m(2x^2 + 2y^2 + 2z^2)$$

$$= 2(\sum m) \cdot (x^2 + y^2 + z^2)$$

$$= 2Ma^2$$

$$\boxed{I = \frac{2}{3}Ma^2} \rightarrow \text{Result}$$

For Hemispherical Shell#

For hemi-spherical shell limits of integration are from 0 to $\pi/2$ and Moment of inertia about symmetry axis OY is

I_{OY}

$$I_{OY} = 2\rho\pi a^4 \int_0^{\pi/2} \sin^3 \theta d\theta$$

$$= \frac{2 \cdot M}{2\pi a^2} \cdot \pi a^4 \cdot \frac{2}{3}$$

$$= \frac{2}{3}Ma^2$$

By Wallis formula

$$\rho = \frac{M}{2\pi a^2}$$

M.I of Hemi-Spherical Spherical Shell

Question# Find the moment of inertia of uniform hemi-spherical shell of mass M , radius a , about

- (i) axis of symmetry (ii) a diameter of plane base
 (iii) an axis through its mass centre and perpendicular to the axis of symmetry.
 (iv) a tangent at the end of symmetry axis

Sol# let O be the centre of the plane base and OA the axis of symmetry. Let G be its mass-centre. Then

$$OG = \frac{1}{2}a$$

Let BD be diameter of plane base perpendicular to the axis of symmetry.

and GC be the axis through G parallel to BD .

Divide the sphere into circular slices perp to symmetry axis and centred at the axis

Consider such a slice of width $a d\theta$.

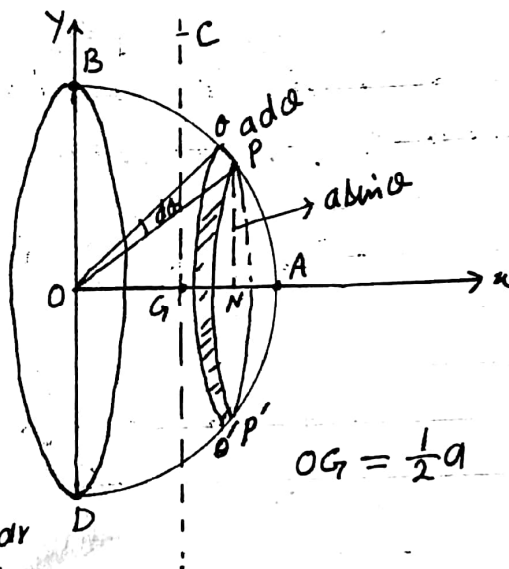
$$\text{Radius of the slice} = PN = a \sin \theta$$

$$\text{Area of the slice} = 2\pi \cdot a \sin \theta \cdot a d\theta$$

$$= 2\pi a^2 \sin \theta d\theta$$

$$\text{Mass of the slice} = \rho \cdot 2\pi a^2 \sin \theta d\theta = dm.$$

The symmetry axis OA of the shell is the centroidal axis of the slice and \perp ar to its plane.



92

Moment of inertia of slice about symmetry axis is

$$dI_{ox} = \text{Mass} \times (\text{radius})^2$$

(By method of uniform ring about centroidal axis perpendicular to the plane of ring)

$$dI_{ox} = 2\pi a^2 \sin\theta \cdot d\theta \cdot a^2 \sin^2\theta$$

$$= 2\pi a^4 \sin^3\theta \cdot d\theta$$

Moment of inertia of shell about symmetry axis is

$$I_{ox} = 2\pi a^4 \int_0^{\pi/2} \sin^3\theta d\theta$$

$$= \frac{2 \cdot M}{2\pi a^2} \cdot \pi a^4 \cdot \frac{2}{3} \quad \text{By Wallis formula}$$

$$I_{ox} = \frac{2}{3} M a^2 \quad \text{which is same}$$

as for spherical shell of radius a .

(ii) About Diameter of Plane Base

The moment of inertia of circular slice about its own diameter cd parallel to diameter BD of plane base is

$$dI_d = \frac{1}{2} (\text{Mass}) \cdot (\text{radius})^2$$

$$= \frac{1}{2} dm \cdot a^2 \sin^2\theta$$

$$dI_d = \frac{1}{2} \cdot (\rho \cdot 2\pi a^2 \sin\theta \cdot d\theta) \cdot (a^2 \sin^2\theta)$$

$$= \rho \pi a^4 \sin^3\theta d\theta$$

Now diameter of slice is its centroidal axis and we may apply parallel axes theorem to find its M.I about BD.

$$dI_{BD} = dI_d + (dm) \cdot (a^2 \cos^2\theta)$$

$$= \rho \pi a^4 \sin^3\theta d\theta + \rho \cdot 2\pi a^2 \sin\theta \cdot d\theta \times a^2 \cos^2\theta$$

$$= \rho \pi a^4 \sin^3\theta d\theta + 2\rho \pi a^4 \sin\theta \cos^2\theta d\theta$$

$$I_{BD} = \rho \pi a^4 \int_0^{\pi/2} \sin^3\theta d\theta + 2\rho \pi a^4 \int_0^{\pi/2} \cos^2\theta \sin\theta d\theta$$

$$= \frac{M}{2\pi a^2} \cdot \pi a^4 \cdot \frac{2}{3} + 2 \cdot \frac{M}{2\pi a^2} \cdot a^4 (-1) \left| \frac{\cos^3\theta}{3} \right|_0^{\pi/2}$$

$$= \frac{1}{3} Ma^2 + \frac{1}{3} Ma^2$$

$$I_{BD} = \frac{2}{3} Ma^2$$

$I_{BD} = \frac{2}{3} Ma^2$

→ Result.

(iii) About Centroidal axis parallel to BD#

The centroidal axis G_C perpendicular to the axis of symmetry will be parallel to BD and at distance $OG_C = \frac{1}{2}a$.
So by parallel axis theorem, we have

$$I_{BD} = I_{GC} + M \cdot OG^2$$

$$\frac{2}{3} Ma^2 = I_{GC} + M \cdot \frac{a^2}{4}$$

$$I_{GC} = \frac{2}{3} Ma^2 - \frac{1}{4} Ma^2$$

$$= Ma^2 \left(\frac{2}{3} - \frac{1}{4} \right)$$

$$I_{GC} = \frac{5}{12} Ma^2 \longrightarrow \text{Result.}$$

(iv) About Tangent at the End of Symmetry Axis #

Tangent at the end A of symmetry axis is parallel to centroidal axis GC and at distance given by

$$GA = OA - OG$$

$$= a - \frac{1}{2}a = \frac{1}{2}a$$

By parallel axes theorem, we have

$$I_{\text{tangent}} = I_{GC} + M \cdot \left(\frac{1}{2}a \right)^2$$

$$= \frac{5}{12} Ma^2 + \frac{Ma^2}{4}$$

$$= Ma^2 \left(\frac{5}{12} + \frac{1}{4} \right)$$

$$= Ma^2 \left(\frac{5+3}{12} \right) = Ma^2 \cdot \frac{8}{12}$$

$$= Ma^2 \cdot \frac{2}{3}$$

$$I_{\text{tangent}} = \frac{2}{3} Ma^2 \longrightarrow \text{Result.}$$

M.I of Uniform Solid Sphere

Question # Find the moment of inertia of a uniform solid sphere about

- (i) # a diameter (or axis of symmetry)
- (ii) # a tangent at any point of sphere.

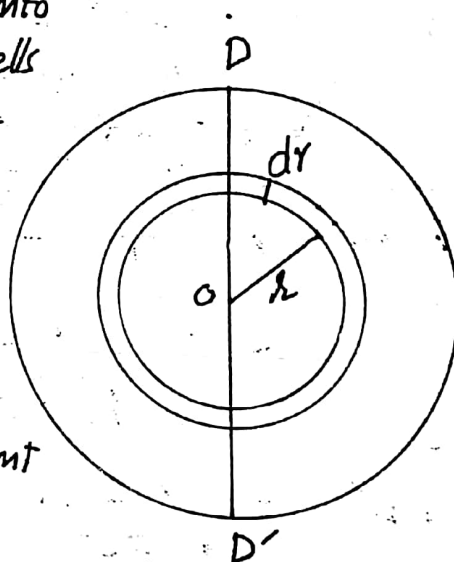
(* By M. Hussain Lecturer (Maths) Govt College Asghar Mall RWP)

Sol # Method-I (Shell Method) #

Consider a sphere of radius a , & mass M . Let O be the centre of sphere and DD' diameter of the sphere.

Divide the sphere into concentric spherical shells centred at the centre of the sphere.

Let r be the radius of one such shell with thickness dr .



Volume of the shell element
 $= dV = \frac{4}{3}\pi r^3$

Area of the shell element $= 4\pi r^2$
 Volume enclosed by the shell
 $= 4\pi r^2 dr$

Mass of the shell $= dm = \rho(4\pi r^2 dr)$
 where ρ is volume density of the sphere

Moment of inertia of shell about its diameter DD' (it will also be diameter of shell)

$$dI_{DD'} = \frac{2}{3}(dm) r^2 \quad (\text{by spherical shell Method})$$

$$dI_{DD'} = \rho \frac{2}{3} (4\pi r^2) dr \cdot r^2$$

$$= \rho \frac{8}{3} \pi r^4 dr$$

Moment of inertia about diameter is

$$I_{DD'} = \rho \frac{8}{3} \pi \int_0^a r^4 dr$$

$$= \rho \frac{8}{3} \pi \left[\frac{r^5}{5} \right]_0^a$$

$$= \rho \cdot \frac{8}{3} \pi \left[\frac{a^5}{5} \right]$$

$$\text{Volume of sphere} = \frac{4}{3} \pi a^3$$

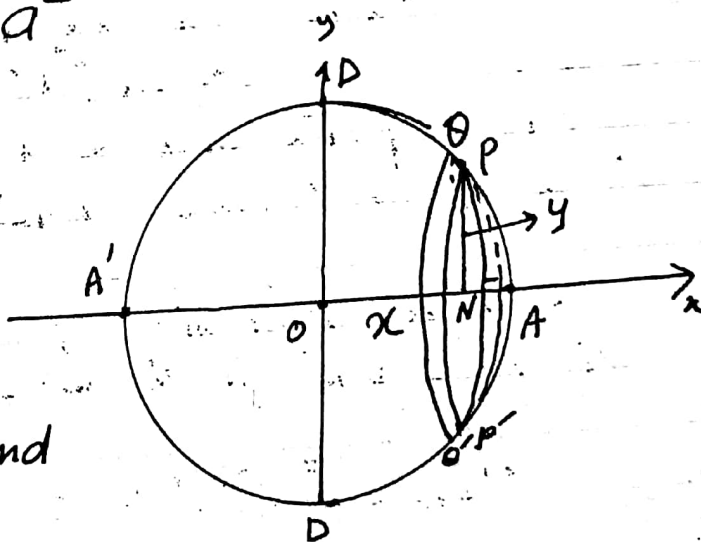
$$\rho = \frac{M}{\frac{4}{3} \pi a^3}$$

$$I_{DD'} = \frac{M}{\frac{4}{3} \pi a^3} \cdot \frac{8}{3} \pi \cdot \frac{a^5}{5}$$

$$I_{DD'} = \frac{2}{5} M a^2$$

Method-II#

Divide the sphere into circular slices (Circular discs) all perpendicular to the diameter OX and centres at Ox



Then $x^2 + y^2 = a^2 - x^2$

Note equation of sphere is
 $x^2 + y^2 + z^2 = a^2$

and circular disc is taken in xy -plane
 let dx be thickness of the disc. Then

Volume occupied by disc $= \pi y^2 dx$
 Mass of the disc is

$$dm = \rho \pi y^2 dx$$

Moment of the disc about ox (centroidal axis perp to the plane of disc) is

$$\begin{aligned} dI_{ox} &= \frac{1}{2} (dm) (\text{radius})^2 \\ &= \frac{1}{2} (\rho \pi y^2 dx) \cdot y^2 \\ &= \frac{1}{2} \rho \pi y^4 dx \\ &= \frac{1}{2} \rho \pi (a^2 - x^2)^2 dx \end{aligned}$$

Hence M.I of the whole sphere about ox is

$$I_{ox} = \frac{1}{2} \rho \pi \int_{-a}^a (a^2 - x^2)^2 dx$$

$$= \frac{1}{2} \rho \pi \cdot 2 \int_0^a (a^2 - x^2)^2 dx$$

(\because integrand is even)

$$I_{xx} = \rho \pi \cdot \int_0^a (a^2 - x^2)^2 dx$$

$$\text{let } x = a \sin \theta \\ dx = a \cos \theta d\theta$$

$$\text{When } x=0 \quad \theta=0 \\ x=a \quad \theta=\pi/2$$

$$I_{xx} = \rho \pi \int_0^{\pi/2} (a^2 - a^2 \sin^2 \theta)^2 \cdot a \cos \theta d\theta$$

$$= \rho \pi a^5 \int_0^{\pi/2} \cos^5 \theta d\theta$$

$$= \rho \pi a^5 \cdot \frac{4 \cdot 2}{5 \cdot 3} \quad \text{By Wallis formula}$$

$$\rho = \frac{M}{\frac{4}{3} \pi a^3}$$

$$I_{xx} = \frac{3}{4} \frac{M}{\pi a^3} \cdot \pi \cdot a^5 \cdot \frac{4 \cdot 2}{5 \cdot 3}$$

$$I_{xx} = \frac{2}{5} M a^2$$

Method-III

$$I_{xx} = \rho \iiint (y^2 + z^2) dV$$

using spherical polar co-ordinates

$$x = r \sin \theta \cos \phi \quad 0 \leq \theta \leq \pi$$

$$y = r \sin \theta \sin \phi \quad 0 \leq \phi \leq 2\pi$$

$$z = r \cos \theta$$

$$I_{xx} = \rho \iiint (r^2 \sin^2 \theta \cos^2 \phi + r^2 \cos^2 \theta) dV \quad dV = r^2 \sin \theta dr d\theta d\phi$$

$$= \rho \int_0^a \int_0^\pi \int_0^{2\pi} (r^2 \sin^2 \theta \cos^2 \phi + r^2 \cos^2 \theta) r^2 \sin \theta dr d\theta d\phi$$

$$= \rho \int_0^a r^4 dr \left[\int_0^\pi \int_0^{2\pi} \sin^3 \theta \cos^2 \phi d\theta d\phi + \int_0^\pi \int_0^{2\pi} \cos^2 \theta \sin \theta d\theta d\phi \right]$$

$$= \frac{\rho a^5}{5} \left[4 \cdot 2 \int_0^{\pi/2} \cos^2 \phi d\phi \cdot \int_0^{\pi/2} \sin^3 \theta d\theta + \left[\phi \right]_0^{2\pi} (-1) \left[\frac{\cos^3 \theta}{3} \right]_0^\pi \right]$$

$$= \frac{\rho a^5}{5} \left[8 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \cdot \frac{2}{3} + 2\pi (-1) \left[-\frac{1}{3} - \frac{1}{3} \right] \right]$$

$$= \frac{\rho a^5}{5} \left[\frac{4}{3} \pi + \frac{4\pi}{3} \right]$$

$$= \rho \cdot \frac{8a^5 \pi}{15}$$

$$= \frac{M}{\frac{4}{3} \pi a^3} \cdot \frac{8a^5 \pi}{15} = \frac{2}{5} M a^2$$

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(ii) About Tangent #

Tangent to sphere will be parallel to diameter of the sphere.

Let LM be tangent of sphere and id, be diameter parallel to the (sphere) tangent. Then by parallel axes Theorem, we have

$$I_{LM} = I_d + Ma^2$$

$$= \frac{2}{5} Ma^2 + Ma^2$$

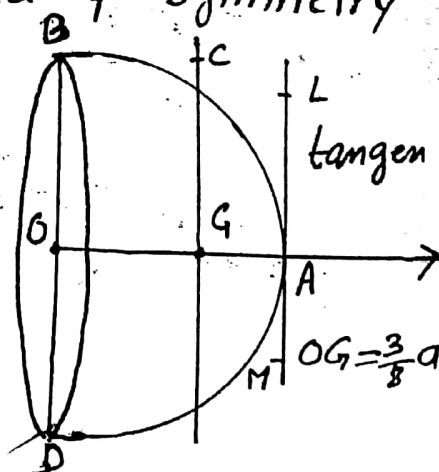
$= \frac{7}{5} Ma^2$
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M.I of Uniform Solid Hemi-sphere #

Question # Find the moment of inertia of uniform hemi-sphere of Mass M, radius (a), about

- (i) # axis of symmetry
- (ii) # a diameter of plane base
- (iii) # an axis through its c-m and perpendicular to the axis of symmetry
- (iv) # a tangent at the end of symmetry axis

Sol # Consider a uniform hemi-sphere of radius a, & mass M. Let G be its centre of mass and GC be centroidal axis perpendicular to the



101

Symmetry axis. Let BD be the diameter of the plane base.

(i) About Axis of Symmetry #

Consider a uniform solid sphere whose centre is at O and whose radius is a . Then

Mass of solid sphere = 2 (Mass of hemi-sphere)

Moment of inertia of the sphere about OA is $= 2M$

$$\frac{2}{5}(2M)a^2$$

Now if we bisect the sphere by a plane through O perpendicular to OA, then contribution of each hemi-sphere is same towards the moment of inertia of the whole sphere.

Hence the moment of inertia of hemi-sphere is half of the m.I of whole sphere.

$$M.I \text{ of hemi-sphere about OA} = \frac{2}{5}Ma^2$$

(ii) Similarly moment of inertia of hemi-sphere about diameter BD is $\frac{2}{5}Ma^2$

(iii) About Centroidal axis perp to OA #

By parallel axes theorem, we have

$$I_{BD} = I_{Gc} + M \cdot OG^2$$

$$\frac{2}{5}Ma^2 = I_{Gc} + M \cdot \left(\frac{3}{8}a\right)^2$$

$$\frac{2}{5}Ma^2 = I_{Gc} + \frac{9}{64}Ma^2$$

$$I_{GC} = \left(\frac{2}{5} - \frac{9}{64} \right) Ma^2$$

$$\boxed{I_{GC} = \left(\frac{83}{320} \right) Ma^2} \longrightarrow \text{Result.}$$

(iv) About Tangent at End A#

Tangent at the end A will be parallel to GC and at distance given by

$$\begin{aligned} GA &= OA - OG \\ &= a - \frac{3}{8}a \end{aligned}$$

$$GA = \frac{5}{8}a$$

By parallel axes theorem.

$$I_{\text{tangent}} = I_{GC} + MGA^2$$

$$= \frac{83}{320} Ma^2 + M \cdot \frac{25}{64} a^2$$

$$= Ma^2 \left(\frac{83}{320} + \frac{25}{64} \right)$$

$$= Ma^2 \left(\frac{83 + 125}{320} \right)$$

$$= Ma^2 \left(\frac{208}{320} \right) = \frac{13}{20} Ma^2$$

$$\boxed{I_{\text{tangent}} = \frac{13}{20} Ma^2} \longrightarrow \text{Result.}$$

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Question # Show that the moment of inertia of a hollow sphere, whose external and internal radii are a , and b , about a diameter axis are given by

$$\frac{2M}{5} \cdot \frac{a^5 - b^5}{a^3 - b^3}$$

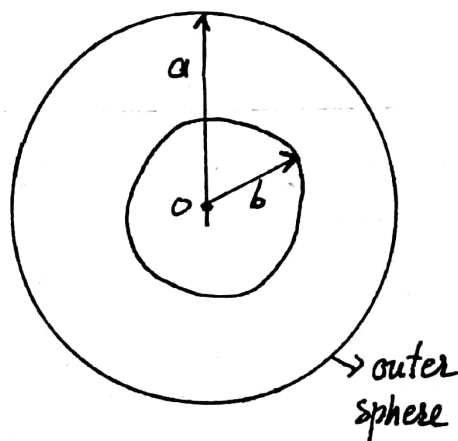
OR

Find the moment of inertia of a spherical shell bounded by two concentric spheres of radii a & b respectively, about diameter.

Sol # The hollow sphere of external and internal radii, a , & b , will be region bounded by two concentric solid spheres of radii a , & b .

Mass of the sphere of radius a , is

$$M_a = \frac{4}{3} \pi a^3 \rho$$



Mass of the sphere of radius b is

$$M_b = \frac{4}{3} \pi b^3 \rho$$

Let M be the mass of hollow sphere. Then

$$M = M_a - M_b = \frac{4}{3} \pi \rho (a^3 - b^3) \rightarrow \textcircled{1}$$

M.I of sphere of radius a , about diameter is

$$I_a = \frac{1}{5} M_a a^2 = \frac{1}{5} \cdot \left(\frac{4}{3} \pi a^3 \rho \right) a^2$$

$$= \frac{4}{15} \pi a^5 \rho$$

$$\text{Similarly } I_b = \frac{4}{15} \pi b^5 \rho$$

Let I be the moment of inertia of hollow sphere.

$$I = I_a - I_b$$

$$= \frac{4}{15} \pi \rho (a^5 - b^5)$$

But $\rho = \frac{3M}{4\pi(a^3 - b^3)}$ by ①

$$\Rightarrow I = \frac{1}{5} \frac{M(a^5 - b^5)}{a^3 - b^3}$$

which is the required result.

برادرز فوٹو سٹاپ

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